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The iteration of cubic polynomials
Part II: Patterns and parapatterns

by

BODIL BRANNER and JOHN H. HUBBARD

The Technical University of Denmark
Lyngby, Denmark

Cornell University
Ithaca, N.Y., U.S.A.

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Chapter 1. Introduction

This is the second in a series of 3 papers devoted to the description of complex cubic polynomials considered as dynamical systems.

From Fatou [F], Julia [J] and Mañe–Sad–Sullivan [MSS], we know that a coarse classification is given by simply counting the number of critical points which escape to infinity. In the previous paper [BH], we identified the topology of the locus of polynomials

$$P_{a,b}(z) = z^3 - 3a^2z + b$$

where at least one critical point escapes. In this paper we will describe how that locus is broken up according to whether one or both critical points escape.

This paper and the previous one give a complete description of the space of cubic polynomials for which one critical point escapes. This is of interest in itself, but should also provide tools for creeping up to the cubic connectedness locus: the locus where neither critical point escapes. More particularly, it should allow us to give a partial description of this locus in terms of stretching rays; this will be the object of the third paper.

1.1. Patterns and their origin in the dynamical plane

As usual in complex analytic dynamics, we plough in the dynamical plane and reap in the parameter space. We will analyze carefully the dynamical plane for cubic polynomials P where at least one critical point escapes. We will do this by building "abstract Riemann surfaces", called patterns, which will turn out to be isomorphic to appropriate domains in the dynamical plane. However, precisely because they have been constructed without any reference to dynamical systems, and in particular are independent of any "infinite process", we can understand them more easily than the dynamical domains they represent.

An outline of the procedure is as follows. Let h_P be the potential defined by

$$h_P(z) = \lim_{n\to\infty} \frac{1}{3^n} \log_+ |P^{\circ n}(z)|$$

as in [DH1, BH], and define the filled-in Julia set as

$$K_P = \{z \in \mathbf{C} \mid h_P(z) = 0\}.$$

For any $z_0 \in \mathbf{C}$ let

$$U_P(z_0) = \{z \in \mathbf{C} \mid h_P(z) > h_P(z_0)\},$$

so that $U_P(z_0) = \mathbf{C} - K_P$ if $z_0 \in K_P$.

Suppose P has a critical point ω_1 escaping to infinity, i.e. $h_P(\omega_1) > 0$, and another critical point ω_2 escaping to infinity more slowly than ω_1 or not at all, i.e. $h_P(\omega_2)<h_P(\omega_1)$. The domain we wish to reconstruct abstractly is $U_P(\omega_2)$; note that $U_P(\omega_2)=\mathbf{C}-K_P$ if ω_2 does not escape. Of course, as a subset of $\mathbf{C}$ these sets are complicated, with boundaries given by transcendental equations if $h_P(\omega_2)>0$, and fractal if $h_P(\omega_2)=0$. They are difficult to "know" in any very precise sense. Forgetting about their embedding in $\mathbf{C}$ and remembering only their structure as Riemann surfaces, they are also fairly complicated, depending on a complex number $\zeta\in\mathbf{C}-\bar{D}$, a real number $h<\log|\zeta|$ and many (sometimes infinitely many) combinatorial data. But for all that they are "knowable", as follows.

The first step is to construct an analytic mapping φ_P defined in a neighborhood of infinity and conjugating P to $z\mapsto z^3$. There are only two such φ_P, and if P is monic there is a unique one which is tangent to the identity at ∞. Now for purely topological reasons, φ_P can be extended to $U_P(\omega_1)$, and $\varphi_P(U_P(\omega_1))=\mathbf{C}-\bar{D}_r$ where D_r is the disc of radius $r=e^{h_P(\omega_1)}$. Thus, as an abstract Riemann surface, $U_P(\omega_1)$ is simply the complement of a disc. The point $P(\omega_1)$ is in $U_P(\omega_1)$, and the number $\varphi_P(P(\omega_1))$ is a dynamical invariant of the polynomial. Call it ζ^3, because we will see in a moment that it has a distinguished cube root.

Observe that $\overline{U_P(\omega_1)}$ is a domain in $\mathbf{C}$ bounded by a curve homeomorphic to a figure eight, with double point the critical point ω_1; and that this curve contains one extra inverse image of $P(\omega_1)$, which we will call the co-critical point ω_1'. The mapping φ_P can be extended to a neighborhood of ω_1', and $\varphi_P(\omega_1')=\zeta$ is the distinguished cube root mentioned above.

From the complex number ζ we can reconstruct $\overline{U_P(\omega_1)}$ as $\mathbf{C}-D_{|\zeta|}$ with the points $j\zeta$ and $j^2\zeta$ identified, where j and j^2 are the non-real cube roots of 1.

Now consider $P^{-1}(\overline{U_P(\omega_1)})$; if $h_P(\omega_2)<\frac{1}{3}h_P(\omega_1)$, the mapping $P\colon P^{-1}(\overline{U_P(\omega_1)})\to\overline{U_P(\omega_1)}$ is a ramified triple covering, ramified at the single point ω_1. There are exactly two such coverings, classified by the component of $\mathbf{C}-\overline{U_P(\omega_1)}$ containing the critical value $P(\omega_2)$.

If $h_P(\omega_2)<\frac{1}{9}h_P(\omega_1)$, the next inverse image $P^{-2}(\overline{U_P(\omega_1)})$ is again specified by the component of $\mathbf{C}-P^{-1}(\overline{U_P(\omega_1)})$ containing $P(\omega_2)$; of course the enumeration of the possible components depends on our previous choice.

This construction of successive inverse images of $\overline{U_P(\omega_1)}$ continues until an inverse image contains $P(\omega_2)$, and can be continued for ever if $\omega_2\in K_P$.

The successive specification of the component of $\mathbf{C}-P^{-n}(\overline{U_P(\omega_1)})$ containing $P(\omega_2)$ is called the combinatorial information. With $\zeta\in\mathbf{C}-\bar{D}$ and $h<\log|\zeta|$ fixed and $h_P(\omega_2)<h$ the structure of the set of values of z such that $h_P(z)>h$ only depends on the combinatorial information; it remains isomorphic to itself as a Riemann surface when P ranges in a component of the set of polynomials defined by $\varphi_P(\omega_1')=\zeta$ and $h_P(\omega_2)<h$.

1.2. Outline of the paper

A *pattern* is a sequence of recursively chosen covering spaces as above. The only analytic information is the number ζ, the remainder of the information is combinatorial, specifying covering spaces in such a way as to capture the "essence" of cubic polynomials. In Chapter 2 of this paper we describe exactly how to build them. It is fairly easy to show that the domains in the dynamical plane described above are isomorphic as Riemann surfaces to appropriate patterns, and this is done in Chapter 3. It is much less obvious that all patterns can be realized in this way.

Patterns come with embedded graphs which divide them up into annuli. The combinatorial description of the covers turns out also to encode the moduli of the annuli, and we are able under appropriate circumstances to estimate these moduli so as to show that components of K_P are points. This, together with the theory of polynomial-like mappings [DH2], leads to quite a precise description of the Julia sets of cubic polynomials for which one critical point escapes. In particular, we give a complete characterization of those polynomials for which the Julia set is a Cantor set. Theorems 5.2 and 5.3 give the precise statements, but the hard work is done in Chapter 4, where a *tableau* is associated to any pattern; it contains enough information to make the estimates on annuli referred to above, but sufficiently little to be reasonably manageable. There the main result is Theorem 4.3.

Curt McMullen proved that those Julia sets to which the tableau argument applies (these are precisely those which are Cantor sets) are of measure zero, and has kindly agreed to our including his result as Theorem 5.9 of our paper.

Next we take up the problem of showing which patterns actually occur. The result is very satisfying: Theorems 8.2 and 9.1 say that every pattern does occur, and tells exactly how two polynomials are related if they have the same pattern.

We could have proved the results of Chapter 5 without ever mentioning patterns, simply by working in the dynamical plane itself. But in Chapters 7, 8 and 9 we see what the drudgery in Chapters 2 and 3 has bought us: a complete abstract description of the *escape locus:* the space of cubic polynomials for which both critical points escape. In Chapter 7 we set up a parameter space for patterns which is itself a complex manifold, called the *parapattern space*. It is universal in the sense that it parametrizes patterns, and there is a map from the escape locus into it, classifying the patterns of polynomials. Our existence and uniqueness statement says that this mapping is an isomorphism: the proof consists essentially in showing that this mapping is analytic and proper.

Further we are able to understand the structure of the locus $\mathscr{B}$ where one critical point escapes and the other does not. It is a fibration over $\mathbf{C}-\bar{D}$. The fiber is made up of uncountably many components, countably many of which are homeomorphic to the Mandelbrot set, i.e. the connectedness locus for quadratic polynomials, and the others are points. The first fact is an application of Mandelbrot-like families of mappings [DH2] and the latter fact (like Theorem 5.2) an application of the tableau argument. Hence we have shown that there are no "queer" components in the fiber. This result can be viewed as an analogue of the following conjecture: the natural map from the Mandelbrot set onto the abstract Mandelbrot set (defined by Thurston) is a bijection; i.e. there are no "queer" inverse images of points. This conjecture is equivalent to the local connectivity conjecture of the Mandelbrot set.

Chapters 10 and 11 are devoted to understanding the topology of the parapattern space. The escape locus, or rather the subset where the critical point $+a$ escapes faster than the other critical point $-a$, is also topologically a fiber bundle over $\mathbf{C}-\bar{D}$, and in Chapter 10 we compute its monodromy. This allows us, at least in principle, to understand the components of the locus where one critical point escapes and the other doesn't. It also allows us to compute the fundamental group of the escape locus, in Chapter 11. The main motivation for doing so is a recent result of Blanchard, Devaney, and Keen [BDK] showing that there is a representation of this fundamental group onto the group of automorphisms of the one-sided 3-shift.

We would like to call the attention of the reader to Chapter 12. This chapter says that most of the constructions of this paper and the previous one go over to polynomials with two critical points of arbitrary degree. But the analogue of Theorem 5.2 on Julia sets *does not go through,* and gives a place to look for possible counterexamples to the generic hyperbolicity conjecture.

Acknowledgements. This paper was almost five years in the writing, and went through many drafts. While (and before) this paper has been written several other papers [Bl1], [Bl2], [BDK], [Br], [L], [M2], [M3] containing studies of cubic polynomials have appeared (at least as preprints) and we have benefited from reading them. In particular we want to emphasize that Pierre Lavaurs [L] settled the conjecture in [BH], showing that the connectedness locus is cellular in all degrees.

During these five years the authors benefited from help, conversations, suggestions and criticism from many people, and hospitality and funding from many institutions. We wish to thank all of them here.

As always in this field, Adrien Douady must be mentioned first: he provided innumerable comments and suggestions without which this paper would be very different, or perhaps would not exist.

Very special thanks are also due to Homer Smith. He made the computer pictures which appear in the paper, and also wrote a number of programs for exploring cubic polynomials, without which the results might never have been found.

Curt McMullen found the proof of Theorem 5.9 and has been so kind as to allow us to include it in our paper. Mitsuhiro Shishikura pointed out that Theorem 4.3(a) failed in higher degrees, correcting an error in an earlier draft of the paper.

We do not know exactly who had the idea of using moduli of annuli to show that some Julia sets are Cantor sets, but it wasn't one of us. We believe that the credit should be shared by Adrien Douady and Paul Blanchard.

We thank Ben Bielefeld, Robert Devaney, Yuval Fisher, Lisa Goldberg, Linda Keen, Pierre Lavaurs, John Milnor, Mary Rees, Dennis Sullivan, Tan Lei and Bill Thurston, for many helpful conversations.

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Chapter 2. Patterns

In this chapter we shall build for each $\zeta \in \mathbf{C} - \bar{D}$ a tree

$$\mathcal{P}(\zeta) = \bigcup_n \mathcal{P}_n(\zeta)$$

of "abstract" Riemann surfaces. This is not very hard once one has the right definitions; none of the proofs are difficult. But it is quite long and cumbersome, especially as the reason for the construction must wait till Chapter 3. These Riemann surfaces are combinatorial reconstructions of domains which arise naturally as subsets of the dynamical plane.

2.1. Topological preliminaries

As indicated in the introduction, our patterns are built from towers of covering spaces. In Section 2.8, we will make bundles out of our patterns; we must therefore think *functorially*, i.e. construct objects not just up to isomorphism but up to *unique* isomorphism. The following two lemmas are topological existence and uniqueness statements; the uniqueness part will allow just that.

LEMMA 2.1. *Let $f: S^1 \to S^1$ be a homeomorphism; then f extends to a homeomorphism $F: \bar{D} \to \bar{D}$ such that $F(0)=0$, and any two such extensions are isotopic among such homeomorphisms.*

Proof. The mapping f can be extended radially. The isotopy statement is proved using the Alexander trick. If F_1 and F_2 are two extensions, set $F = F_1 \circ F_2^{-1}$. Then the family

$$F(t,x) = \begin{cases} x & \text{if } \|x\| > t \\ tF(x/t) & \text{if } \|x\| \leq t \end{cases}$$

provides an isotopy of F to the identity. Q.E.D. of Lemma 2.1

LEMMA 2.2. *Let a and b be two points in $X = \mathbf{C}$. Then there exists a connected triple cover $\pi: Y \to X$ ramified above a and b such that $\pi^{-1}(a)$ and $\pi^{-1}(b)$ both consist of precisely two points, and if $\pi_Y: Y \to X$ and $\pi_Z: Z \to X$ are two such covers, there exists a unique covering homeomorphism $Y \to Z$.*

Figure 2.1 illustrates such a ramified covering, as well as the notation used in the proof.

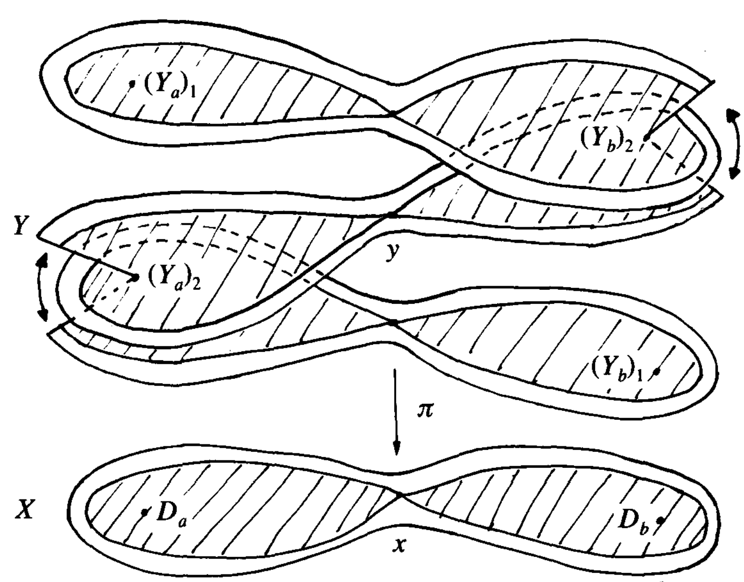


Fig. 2.1

Proof. The cubic polynomial

$$P(z)=\frac{b-a}{4}(z^3-3z)+\frac{a+b}{2}$$

has the property that $P\colon \mathbf{C}\to\mathbf{C}$ is a connected triple cover ramified above a and b with $P^{-1}(a)=\{1,-2\}$ and $P^{-1}(b)=\{-1,2\}$. This establishes existence.

The uniqueness and rigidity can be seen as follows. Let $D_a, D_b\subset X$ be closed discs containing a and b respectively in their interior, and with $D_a\cap D_b=\{x\}$ a single point. Let $\pi_Y\colon Y\to X$ and $\pi_Z\colon Z\to X$ be two ramified triple covers as in 2.2. Define $Y_a=\pi_Y{}^{-1}(D_a)$, etc. Our hypotheses imply that each of Y_a, Y_b, Z_a, Z_b consists of two discs $(Y_a)_1$, $(Y_a)_2$, etc. such that the first covers its image with degree 1 and the second with degree 2. The set $(Y_a)_2\cap(Y_b)_2$ consists of a single point y; indeed it cannot be empty because both $Y_a\cap\pi_Y{}^{-1}(X)$ and $Y_b\cap\pi_Y{}^{-1}(X)$ consists of two points, and $\pi_Y{}^{-1}(X)$ only has three points; and it cannot contain two points for otherwise Y would be disconnected. Similarly $(Z_a)_2\cap(Z_b)_2=\{z\}$.

There now exists a unique covering homeomorphism $(Y_a)_1\to(Z_a)_1$ since these are simple covers, and a unique covering homeomorphism $(Y_a)_2\to(Z_a)_2$ mapping y to z; and similarly with a replaced by b. These four homeomorphisms piece together to give a covering homeomorphism of $\pi_Y{}^{-1}(D_a\cup D_b)$ to $\pi_Z{}^{-1}(D_a\cup D_b)$. Since the inclusion of $D_a\cup D_b$ into X is a homotopy equivalence, this homeomorphism extends uniquely to a covering homeomorphism $Y\to Z$.

The rigidity follows also: any covering transformation has to map y to z, so the covering homeomorphism is unique. Q.E.D. for Lemma 2.2

Definition 2.3. The triple cover described by Lemma 2.2 will be called the *standard triple cover.*

2.2. Constructing the tree of patterns

Choose a complex number ζ with $|\zeta|=r>1$. We shall recursively construct for each $N\in\mathbf{Z}$

(a) a finite set $\mathcal{P}_N(\zeta)$ of Riemann surfaces, called *ζ-patterns of depth N,* and

(b) a mapping $s_N\colon \mathcal{P}_N(\zeta)\to\mathcal{P}_{N-1}(\zeta)$.

Each $R\in\mathcal{P}_N(\zeta)$ will come with

(c) an embedding into a simply connected Riemann surface $\bar{R}$ isomorphic to $\mathbf{C}$;

(d) if $N>0$ a distinguished point $\omega_R\in R$;

(e) a triple covering map $\pi_R\colon R\to s_N(R)$ ramified at ω_R (i.e. unramified if $N\leq 0$);

(f) an inclusion $i_R\colon s_N(R)\to R$, satisfying $i_{s_N(R)}\circ\pi_{s_N(R)}=\pi_R\circ i_R$; and
(g) a collection of open annuli $B_n(R)\subset R$, $n<N$, nested in $\bar{R}$, such that

$$i_R(B_n(s_N(R)))=B_n(R) \quad \text{for} \quad n<N-1.$$

Construction of $\mathcal{P}_N(\zeta)$, $N\leq 0$. The set $\mathcal{P}_N(\zeta)$ consists for each $N\leq 0$ of the single Riemann surface $R_N=\mathbf{C}-\bar{D}_{r^{3^{-N}}}$, and $\bar{R}_N=\mathbf{C}$. All maps $\pi_{R_N}\colon R_N\to R_{N-1}$ are given by $z\mapsto z^3$, and the mappings $i_{R_N}\colon R_{N-1}\to R_N$ are the inclusions. Finally set

$$B_n(R_N)=\{z\,|\,r^{3^{-n}}<|z|<r^{3^{-n+1}}\}.$$

Construction of $\mathcal{P}_1(\zeta)$. Again $\mathcal{P}_1(\zeta)$ consists of just one Riemann surface R_1. Using Lemma 2.2, there exists a unique standard triple cover

$$\pi_{\bar{R}_1}\colon\bar{R}_1\to\bar{R}_0$$

of $\mathbf{C}=\bar{R}_0$ ramified above ζ^3 and 0; let $R_1=(\pi_{\bar{R}_1})^{-1}(R_0)$ and $\pi_{R_1}\colon R_1\to R_0$ be the restriction of $\pi_{\bar{R}_1}$. Further let ω_{R_1} be the critical inverse image of ζ^3, and ω'_{R_1} be the other (co-critical) inverse image. This specifies parts (a)–(e) of the construction, for $n=1$. Note that so far we have only used ζ^3, and not ζ.

For part (f), observe that

— the mapping $z\mapsto z^3$ makes R_0 into a connected triple cover of R_{-1}, and

— the restriction of $\pi_{R_1}\colon R_1\to R_0$ to $(\pi_{R_1})^{-1}(R_{-1})$ is also a connected triple cover of R_{-1}.

Therefore, there are three covering isomorphisms

$$j\colon R_0\to(\pi_{R_1})^{-1}(R_{-1}).$$

Precisely one of these satisfies $\lim_{z\to\zeta}j(z)=\omega'_{R_1}$. This is the mapping $i_{R_1}\colon R_0\to R_1$.

Define $B_0(R_1)=i_{R_1}(\{z\,|\,r<|z|<r^3\})$. See Figure 2.2.

Construction of $\mathcal{P}_N(\zeta), N>1$. The elements of $s_{N+1}^{-1}(R)$ will be indexed by the bounded components of $R-\bar{B}_{N-1}(R)$. We assume by induction that each such component A is an annulus which shares a boundary with a unique component D_A of $\bar{R}-R$, homeomorphic to a disc. Pick a point $x\in D_A$. Let $\pi_{\bar{R}_A}\colon\bar{R}_A\to\bar{R}$ be the standard triple cover of $\bar{R}$ ramified above x and $i_R(\pi_R(\omega_R))$; let $R_A=(\pi_{\bar{R}_A})^{-1}(R)$ and π_{R_A} be the restriction of $\pi_{\bar{R}_A}$ to R_A. (Figure 2.3 shows construction of $\mathcal{P}_2(\zeta)$ from $\mathcal{P}_1(\zeta)$.)

We still need to construct the inclusion $i_{R_A}\colon R\to R_A$. The surface R was constructed from $s_N(R)$ using the choice of a point $y\in\overline{s_N(R)}$; extend $i_R\colon s_N(R)\to R$ to a homeomor-

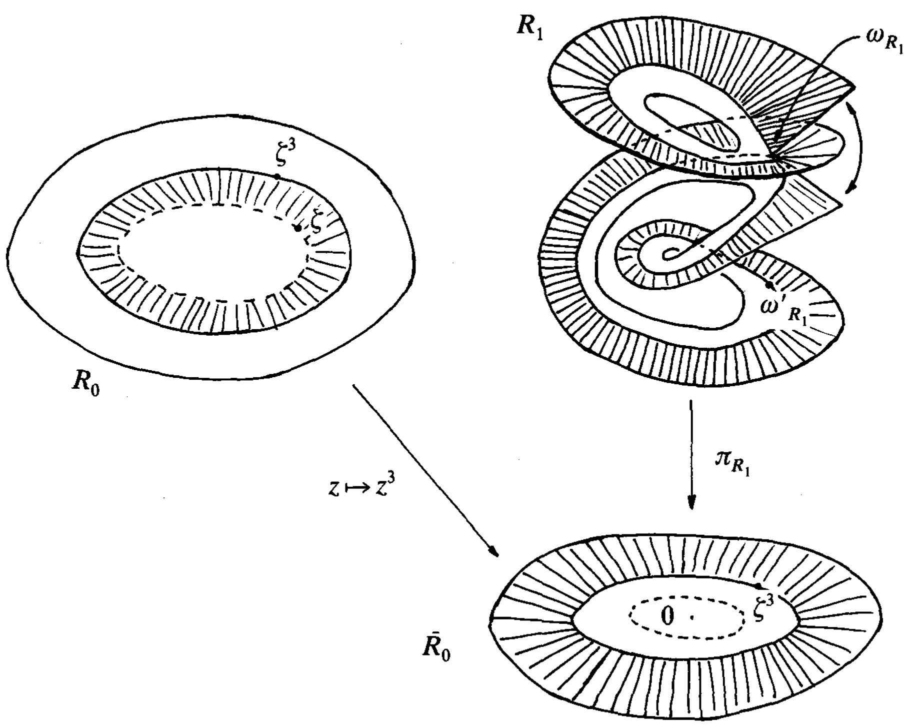


Fig. 2.2

phism $\bar{i}_R: \overline{s_N(R)} \to \bar{R}$ so that $\bar{i}_R(y)=x$. Such an extension exists by Lemma 2.1, and all such extensions are isotopic.

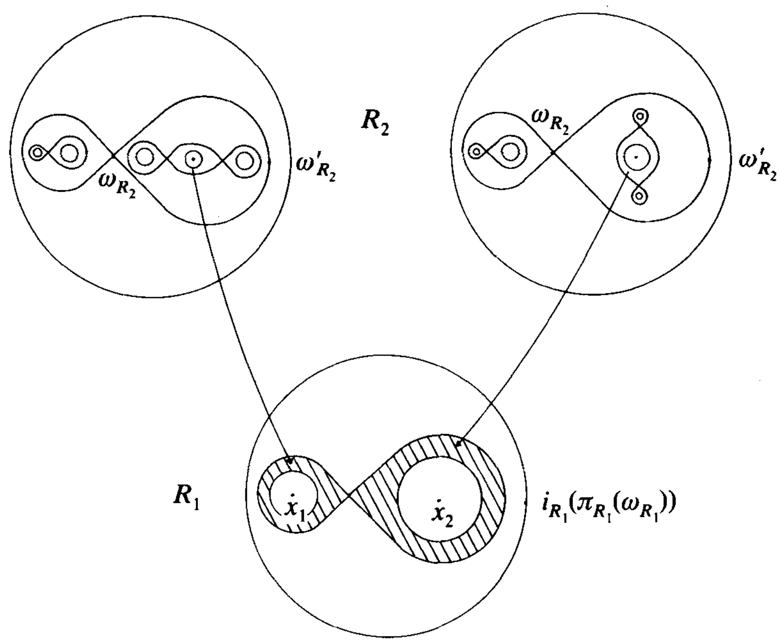
By the functoriality of the triple cover (Lemma 2.2) the mapping $\bar{i}_R$ lifts uniquely to a homeomorphism $\bar{R} \to \bar{R}_A$. The inclusion i_{R_A} is the restriction of this lifting to R. It is clearly independent of the extension $\bar{i}_R$ chosen. The annulus $B_N(R_A)$ is $i_{R_A}(A)$.

Remark 2.4 (a) The construction of $\mathcal{P}_N(\zeta)$ for $N=1$ is essentially a special case of the construction for $N>1$. Only the definition of i_R was different.

(b) The construction of $\bar{R}_A$ from R depends not only on A but also on the choice of a point $x \in D_A$. It is easy to check, however, that neither R_A, π_{R_A}, or i_{R_A} depends on this choice (up to unique isomorphism of Riemann surfaces).

2.3. The potential function h_R

For any $R \in \mathcal{P}_N(\zeta)$, there is a unique harmonic function $h_R: R \to \mathbf{R}_+$ which "extends" the function $\log|z|$. It is defined recursively on patterns of increasing depth by setting



The standard triple cover ramified above $i_{R_1}(\pi_{R_1}(\omega_{R_1}))$ and $x_1 \in D_1, x_2 \in D_2$ respectively. The arrow is drawn from an annulus of R to an annulus of $R_1 = s_2(R)$. The annulus of R is the unique annulus at level 2 which double covers an annulus of R_1.

Fig. 2.3

$$h_R(z) = \frac{1}{3} h_{s_N(R)}(\pi_R(z)),$$

and $h_R(z) = \log|z|$ when $N \leq 0$.

This function satisfies $h_R(i_R(z)) = h_{s_N(R)}(z)$. Use this function to define subsets

$$R_n = \left\{ z \in R \,\middle|\, h_R(z) > \frac{\log|\zeta|}{3^n} \right\};$$

we see from the two formulas above that i_R induces an isomorphism of the pattern $s_N(R)$ with R_{n-1}. Thus each R_n is canonically isomorphic to a ζ-pattern in $\mathcal{P}_n(\zeta)$, allowing us to think of a pattern as an increasing union of Riemann surfaces with appropriate maps; we will frequently allow ourselves to speak of $s_N(R)$ as a subset of R.

2.4. Critical graphs, annuli and arguments

The critical values of h_R are the numbers $(\log r)/3^n$, for $0 \leq n < N$, where $r = |\zeta|$. Let

$$\Gamma_n(R) = \left\{ z \in R \,\middle|\, h_R(z) = \frac{\log r}{3^{n-1}} \right\} \quad \text{for} \quad n \leq N;$$

for $n \leq 0$, $\Gamma_n(R)$ is a simple closed curve, but for $n>0$, $\Gamma_n(R)$ are graphs with double points. Further set

$$\Gamma(R) = \bigcup_{n \leq N} \Gamma_n(R).$$

The number of critical points of h_R belonging to $\Gamma_n(R)$ is 3^{n-1}, since there is one critical point ω_R on $\Gamma_1(R)$ and for $n \geq 2$ a point $x \in \Gamma_n(R)$ is critical if and only if $\pi_R(x) \in \Gamma_{n-1}(R)$ is critical.

Define the *annuli* of R to be the connected components of $R - \Gamma(R)$. The annuli in $R_n - R_{n-1}$ are said to be at *level n*. At each positive level n there is exactly one annulus $C_n(R)$ which is mapped by π_R with degree 2, and its image is $B_{n-1}(R)$; all other annuli of positive level are mapped by π_R with degree 1 (cf. Figure 2.3). The annulus $C_n(R)$ is called *the critical annulus* of R at level n, and $B_{n-1}(R)$ *the critical value annulus* of R at level $n-1$.

The mapping $\pi_R\colon \Gamma_{n+1}(R) \to \Gamma_n(R)$ is a triple cover for any $n>1$. One component $\Gamma'_n(R)$ of $\Gamma_n(R)$ is special, *the critical value graph at level n,* the one surrounded by the critical value annulus $B_{n-1}(R)$. The inverse image $\pi_R^{-1}(\Gamma'_n(R))$ has two components, while for all other components Λ of $\Gamma_n(R)$ the inverse image $\pi_R^{-1}(\Lambda)$ has three components. Since $\Gamma_1(R)$ is a figure eight and therefore has just one component, it is easy to find the number of components of $\Gamma_n(R)$ to be $(1+3^{n-1})/2$ for $n \geq 1$ (see Figures 2.3 and 2.5).

For any $R \in \mathcal{P}_n(\zeta)$, the critical graph $\Gamma_0(R)$ is always the same: it is the circle of radius $|\zeta|^3$. Therefore we can call it simply Γ_0, and use it to compare *arguments* of points even if they lie in different patterns of $\mathcal{P}_n(\zeta)$. The *argument* $\arg(z)$ of a point $z \in \Gamma_0$ is $\theta \in \mathbf{R}/\mathbf{Z}$ where $z = r^3 e^{2\pi i\theta}$. The set of *arguments* $\arg(z) \subset \mathbf{R}/\mathbf{Z}$ *of a point* $z \in R$ is the set of arguments of points of Γ_0 at which the (possibly broken) ascending rays from z meet Γ_0. There is a unique argument if the ascending ray contains no critical point of h_R. If the ascending ray from z meets such a critical point x then there are several ways of continuing it with ascending rays from x. The argument of the intersection of all such ascending rays with Γ_0 are then the arguments of z.

For any component Λ of $\Gamma_n(R)$, define the *arguments of* Λ,

$$\arg(\Lambda) = \bigcup_{z \in \Lambda} \arg(z) \subset \mathbf{R}/\mathbf{Z}.$$

Clearly $\arg(\Lambda)$ is for each Λ some finite union of closed intervals, and these form a cover of $\mathbf{R}/\mathbf{Z}$ and are disjoint except at their endpoints as Λ runs through all components of $\Gamma_n(R)$ for any fixed $n \geq 1$.

The following proposition sounds analogous but isn't really: it is a "parameter statement"; it will be essential in Chapter 11.

PROPOSITION 2.5. *For any $N \geq 1$, the sets* $\arg(\Gamma'_N(R))$ *for $R \in \mathcal{P}_N(\zeta)$ form a covering of the circle $\mathbf{R}/\mathbf{Z}$ by closed intervals, disjoint except at their endpoints. These endpoints are precisely the arguments of all critical points of all h_R which lie in $(\Gamma'_n(R)$ for some $R \in \mathcal{P}_n(\zeta)$ and some n with $1 \leq n < N$.*

Proof. The proof, as always, goes by induction on N. It is clear if $N=1$, since in that case there is only one pattern $R \in \mathcal{P}_1(\zeta)$, and the graph $\Gamma_1(R) = \Gamma'_1(R)$ has only one component.

Assume by induction that the result is true for all $M < N$. For each $R \in \mathcal{P}_{N-1}(\zeta)$ we have that

$$\bigcup_{s_N(R')=R} \arg(\Gamma'(R')) = \arg(\Gamma'(R))$$

since each annulus inside $\Gamma'(R)$ is the critical value annulus $B_N(R')$ for precisely one $R' \in s_N^{-1}(R)$. Q.E.D. for Proposition 2.5

When drawing an element $R \in \mathcal{P}_n(\zeta)$ the main objects represented are the critical graphs $\Gamma_n(R)$ together with the gradient curves of h_R emanating from the critical points of h_R. The insight into the structure of such Riemann surfaces R came to the authors largely by drawing lots and lots of graphs.

2.5. The tree of real patterns

Choose $r > 1$ real, and consider the case $\zeta = r$.

PROPOSITION 2.6. *Suppose $R \in \mathcal{P}_N(r)$, $r > 1$. Each critical point of the potential function $h_R: R \to \mathbf{R}_+$ has precisely two ascending rays, which contain no further critical points and intersect the circle $\Gamma_{-1}(R)$ at points with arguments $p_1/3^k$ and $p_2/3^k$ for some integers p_1 and p_2 not divisible by 3. Moreover all numbers $p/3^k$ for $p = 1, \ldots, 3^k - 1$ are obtained this way.*

Remark 2.7. The statement is true because we are speaking of a pattern in $\mathcal{P}(r)$ with r real. It can fail for other values of ζ. For instance, for $\zeta = re^{2\pi i/3}$, a pattern R in $\mathcal{P}_N(\zeta)$ for $N > 1$ looks precisely like Figure 2.4 (a) or (b) down to level 2; in particular ascending rays emanating from some critical points of h_R which lie in $\Gamma_2(R)$ meet the critical point ω_R.

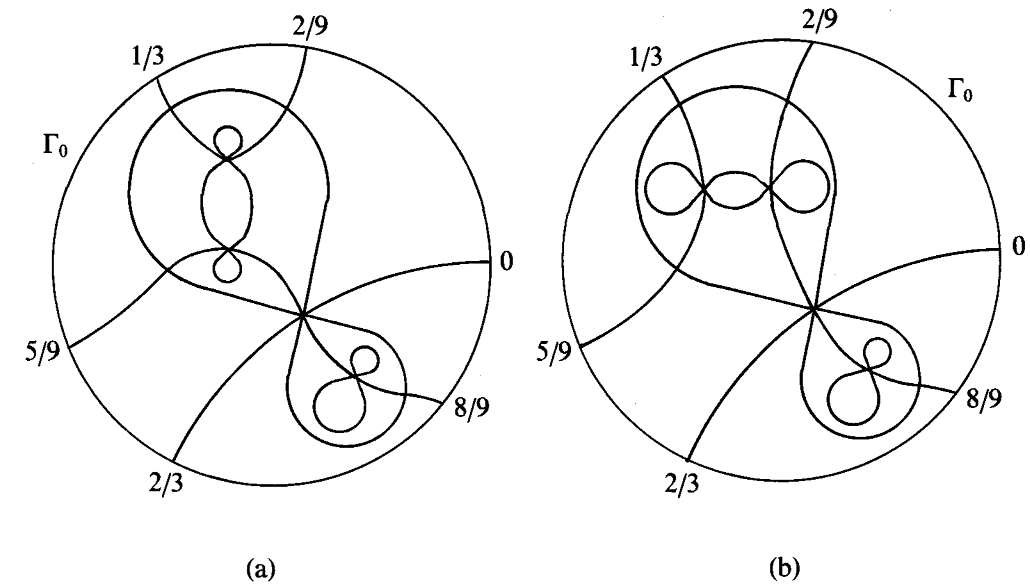


Fig. 2.4

This occurs because the arguments of the ascending rays emanating from ω_R are 0 and 2/3 (the ray of argument 1/3 leads to the co-critical point). The thirds of these two angles are {0, 1/3, 2/3} and {2/9, 5/9, 8/9} respectively. Two of these thirds coincide with the angles already used. The point of the proof is to show that if we choose ζ real, this phenomenon does not occur, and thirds of an argument leading to a critical point at level N do not coincide with arguments of rays leading to critical points at lower levels.

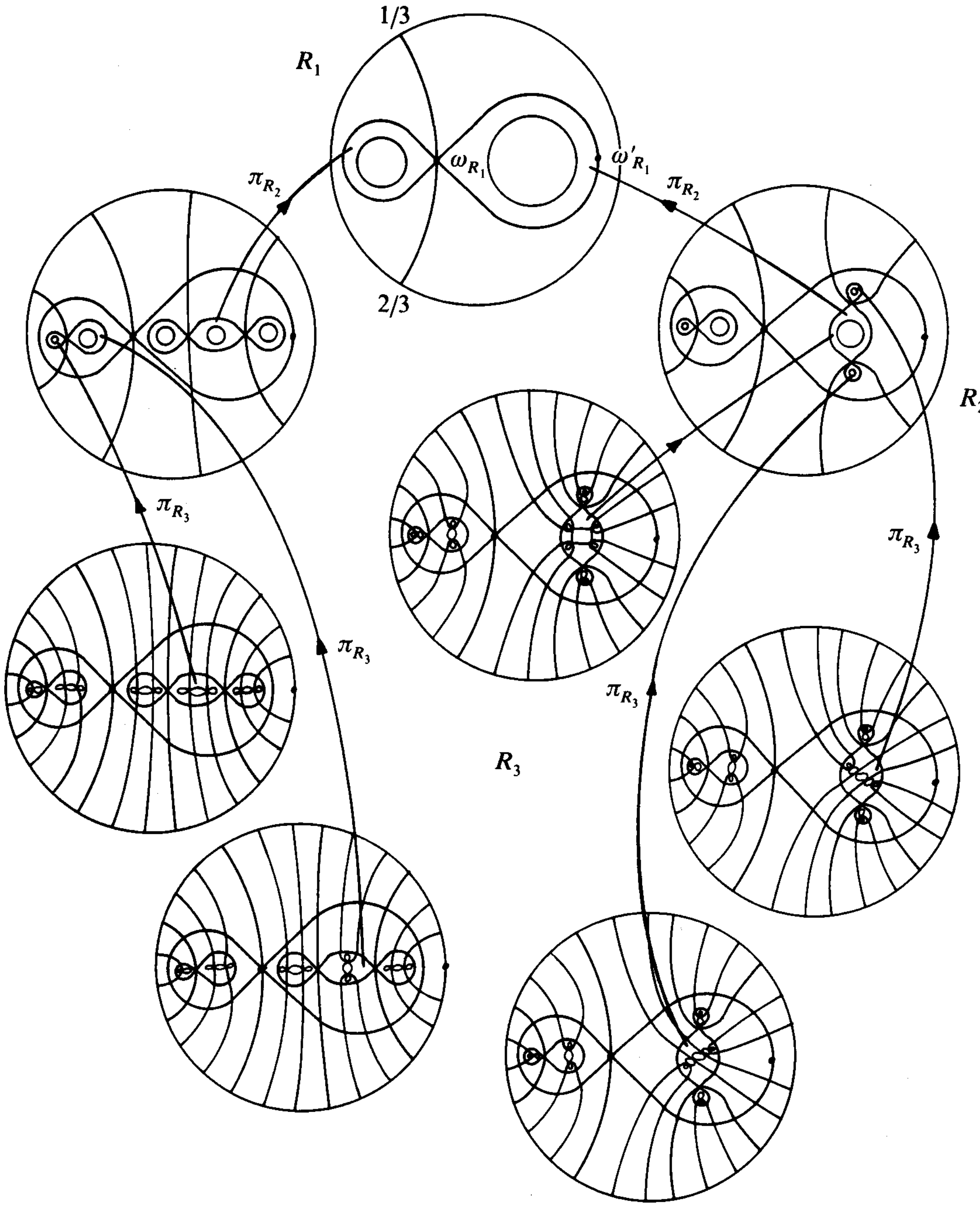
Figure 2.5 represents $\mathscr{P}_N(\zeta)$ for ζ real and positive, and $N=1,2,3$. The arrows associate to each $R\in\mathscr{P}_N(r)$ the annulus A of $s_N(R)$ which labels it, i.e. such that $(s_N(R))_A=R$.

Proof of Proposition 2.6. First observe that for each $k\geq 1$ the number of rationals of the form $p/3^k$ with $p=1,\ldots,3^k-1$ and p and 3 relatively prime is $2\cdot 3^{k-1}$.

Moreover, each rational of the form $p/3^k$ with $p=1,\ldots,3^k-1$ and $(p,3)=1$ has the following preimages under multiplication by 3 in $\mathbf{R}/\mathbf{Z}$:

$$\frac{p}{3^{k+1}},\quad \frac{p+3^k}{3^{k+1}},\quad \frac{p+2\cdot 3^k}{3^{k+1}};$$

i.e. of the same form.



An arrow is drawn from the critical annulus at level N of $R \in \mathscr{P}_N(r)$ to the annulus A of $s_N(R)$ such that $i_R(A) = B_{N-1}(R)$.

Fig. 2.5

Recall from Section 2.4 that the number of critical points of h_R belonging to $\Gamma_n(R)$ is 3^{n-1}. We will prove the result by induction and start the induction by observing that there are precisely two ascending rays emanating from ω_R with arguments 1/3 and 2/3.

The inductive hypotheses: The ascending rays emanating from critical points in $\Gamma_n(R)$ intersect $\Gamma_{-1}(R)$ at points of arguments $p/3^n$ with $p=1,\dots,3^n-1$ and $(p,3)=1$. In particular they do not contain the critical value $\pi_R(\omega_R)=r^3$ of π_R.

The inverse image of two ascending rays emanating from a critical point of h_R in $\Gamma_n(R)$ consists of 6 ascending rays emanating in pairs from 3 critical points in $\Gamma_{n+1}(R)$. From the count above, we see that the rays with arguments $p'/3^{n+1}$ with $p'=1,\dots,3^{n+1}-1$ and $(p',3)=1$ account for all such inverse images.

Q.E.D. for Proposition 2.6

2.6. Patterns of infinite depth

A *ζ-pattern of infinite depth* is a sequence $(R_n)_{-\infty<n<\infty}\in\mathcal{P}_\infty(\zeta)$ with each $R_n\in\mathcal{P}_n(\zeta)$ and $s_n(R_n)=R_{n-1}$.

We can associate to each infinite pattern the Riemann surface

$$R_\infty=\varinjlim(R_n, i_{R_{n+1}}),$$

and since the critical points correspond under the inclusions, R_∞ has a distinguished point ω_{R_∞}. The mapping $\pi_{R_\infty}:R_\infty\to R_\infty$ induced by the maps π_{R_n} is a triple covering map ramified at ω_{R_∞}.

The potential functions h_{R_n} are also compatible under the inclusions and induce a harmonic function $h_{R_\infty}:R_\infty\to\mathbf{R}_+$ satisfying $h_{R_\infty}(z)=\frac{1}{3}h_{R_\infty}(\pi_{R_\infty}(z))$.

Given the Riemann surface R_∞ we can again define the subsets

$$R_n'=\left\{z\in R \,\middle|\, h_{R_\infty}(z)>\frac{\log|\zeta|}{3^n}\right\}.$$

Each R_n' is canonically isomorphic to the ζ-pattern $R_n\in\mathcal{P}_n(\zeta)$, so we shall identify them.

2.7. Pattern isomorphisms

Let R and R' be two patterns of depth $N\leqslant\infty$. An *isomorphism* $f:R\to R'$ is a collection of analytic isomorphisms

$$f_n:R_n\to R_n',\quad n\leqslant N,$$

conjugating all π_{R_n} to $\pi_{R_n'}$ and i_{R_n} to $i_{R_n'}$.

This definition is actually much too strong if $1<N\leq\infty$: all the structure of the pattern up to the determination of $\pm\zeta$ is already encoded in the analytic structure of the Riemann surface R as the Propositions 2.8 and 2.9 will show.

PROPOSITION 2.8. *If R and R' are two patterns of depth N and N' respectively, $1<N$, $N'\leq\infty$ and if $F: R\rightarrow R'$ is an analytic isomorphism, then $N=N'$ and the family of maps $f=(f_n)_{n\leq N}$ given by $f_n=F|_{R_n}$ is an isomorphism of patterns.*

Proof. The proof consists of showing that all the pattern structure, except the number ζ which we will deal with below, can be reconstructed from simply the Riemann surface.

If depth$(R)=N$ is finite then the function h_R is a harmonic function on R having a logarithmic pole at ∞ and such that $h_R(z)\rightarrow(\log|\zeta|)/3^N$ as z tends to the other boundary components of R. By the maximum principle, this shows that h_R is determined by the analytic structure of R, the choice of the end ∞ and the number $r=|\zeta|$. Since the depth of the pattern R is the number of critical values of h_R, we see that N is also determined by the data above.

The end ∞ is the unique puncture of R, so it depends only on the analytic structure of R.

Reconstructing the number r is a bit trickier. Let g be the harmonic function on R having a logarithmic pole at ∞ and tending to 0 on the other boundary components. By the maximum principle for harmonic functions, $g=h_R+C$ for some constant C, so h_R and g have the same level curves.

If $a_0>a_1>\dots$ are the critical values of g, then

$$R_n=\{z\in R|\, g(z)>a_n\}.$$

The region $R_1-\mathrm{clos}(R_0)$ consists of two annuli A_1, A_2 with moduli M_1, M_2 satisfying $M_1=2M_2$, and r is determined by the equation $M_1=(\log r)/\pi$.

The mappings $i_{R_n}: R_{n-1}\rightarrow R_n$ are simply the inclusions.

Finally we reconstruct the mappings $\pi_{R_n}: R_n\rightarrow R_{n-1}$ as follows: the harmonic function h_R has a unique critical point ω_R at level $\log r$ and a unique critical point ξ at level $\log r/3$ in the closure of A_1. Set $A_0=R_0-\mathrm{clos}(R_{-1})$ where $R_{-1}=\{z\in R_N|\, h_R(z)>3\log r\}$. The subset A_0 is an annulus with modulus M_1 so there exists a unique isomorphism of A_1 onto A_0 mapping ξ to ω_R. This mapping must therefore coincide with π_{R_n} for any $1<n\leq N$. Therefore π_{R_n} is defined by analytic continuation.

If depth(R) is infinite then the point ∞ is the only isolated end of the Riemann surface and h_{R_∞} is the unique harmonic function having a logarithmic pole at ∞ and tending to 0 at the other ends. Exactly as for a pattern of finite depth we can construct the number $r=\log|\zeta|$, the subsets R_n, the point ω_{R_∞} and the mappings i_{R_n} and π_{R_n}.

Q.E.D. for Proposition 2.8

If R is a ζ-pattern of depth N with $N>1$, then the number ζ is not determined by the Riemann surface R, but ζ^2 is. The mapping $i_R\circ\pi_R$ is an analytic mapping which has at ∞ a fixed point which is simultaneously a critical point of degree 3, hence $i_R\circ\pi_R$ is conjugate to $z\mapsto z^3$. There are precisely two conjugating maps, since $z\mapsto -z$ conjugates $z\mapsto z^3$ to itself and is the only analytic map other than the identity to do so.

Either conjugating map extends to R_0, and this extension has a limit at the co-critical point. The two limits are opposite of each other, and their square is ζ^2.

Proposition 2.9 shows that the ambiguity is real.

PROPOSITION 2.9. *There exists a mapping*

$$\tau\colon \mathcal{P}(\zeta)\to\mathcal{P}(-\zeta),$$

and for each $R\in\mathcal{P}_N(\zeta)$ *an analytic isomorphism*

$$\tau_R\colon R\to\tau(R)$$

such that if $f=(f_n\colon R_n\to R'_n)_{n\le N}$ *is an isomorphism of patterns, then either* $R_n=R'_n$ *and* $f_n=\mathrm{id}$ *for all n, or* $R'_n=\tau(R'_n)$ *and* $f_n=\tau_{R_n}$ *for all n.*

Proof. We construct τ and the τ_R, as usual, by induction. Start the induction by setting $\tau(R_0)$ to be the unique element $R'_0\in\mathcal{P}_0(-\zeta)$, and defining $\tau_{R_0}\colon R_0\to R'_0$ by $\tau_{R_0}(z)=-z$. Note that τ_{R_0} does conjugate π_{R_0} to $\pi_{R'_0}$.

Suppose by induction that τ and the τ_R have been defined for all $m<n$. Choose $R\in\mathcal{P}_n(\zeta)$, and let $S=s_n(R)$, $S'=\tau(S)\in\mathcal{P}_{n-1}(-\zeta)$. If A is the annulus of $S-\mathrm{clos}(i_S(s_{n-1}(S)))$ defining R, then $A'=\tau_S(A)$ defines an element $R'\in\mathcal{P}_n(-\zeta)$. The mapping τ_S extends to a homemorphism $\tau_{\bar S}\colon\bar S\to\bar S'$, which can by Lemma 2.1 be chosen to send ramification values to ramification values. Since $\bar R$ and $\bar R'$ are standard triple covers of $\bar S$ and $\bar S'$ respectively, the mapping τ_S lifts to an isomorphism $\tau_R\colon R\to R'$. This constructs τ and all the τ_R.

Note that the construction of τ was "unique": given that $\tau(S)=S'$, there is a unique $R'\in s_n^{-1}(S')$ such that there exists an extension of $\tau_S\colon S\to S'$ to $\tau_R\colon R\to R'$, and the extension is itself unique.

Now suppose that $f=(f_n\colon R_n\to R'_n)_{n\le N}$ is an isomorphism of patterns. Then $f_0\colon R_0\to R'_0$ must be either $z\mapsto z$ or $z\mapsto -z$ in order to conjugate π_{R_0} to $\pi_{R'_0}$ since these are the only automorphisms of a neighborhood of ∞ in $\mathbf{C}$ to commute with $z\mapsto z^3$.

The result now follows from the uniqueness inherent in the construction of τ, which itself follows from the uniqueness of Lemma 2.2. Q.E.D. for Proposition 2.9

COROLLARY 2.10. *If $f: R_\infty \to R'_\infty$ is an isomorphism of patterns of infinite depth, then either $R_\infty = R'_\infty$ and $f = \mathrm{id}$ or $R'_n = \tau(R'_n)$ and $f|_{R_n} = \tau_{R_n}$ for all n. In that case, we will say that $\tau(R_\infty) = R'_\infty$.*

Remark 2.11. We can also read ζ^2 from a pattern R of depth $N>1$ without looking near ∞, as follows. From the proof of Proposition 2.8 we know how to compute $r=|\zeta|$. Choose a conformal map

$$\psi: \{z \in \mathbf{C} \mid r < |z| < r^3\} \to A_1$$

which extends to the boundary such that inner boundary is mapped to inner boundary; such a ψ is uniquely determined up to rotation. There exists two points ξ_1 and ξ_2 with $\psi(\xi_1)=\psi(\xi_2)=\xi$, and a unique point ω such that $\psi(\omega)=\omega_R$. One can check that $\xi_1^3=\xi_2^3$, so there exists a unique third point ξ_3 such that $\xi_3^3=\xi_1^3=\xi_2^3$. The ratio ω/ξ_3 does not depend on the choice of ψ, and is ζ^2. See Figure 2.6.

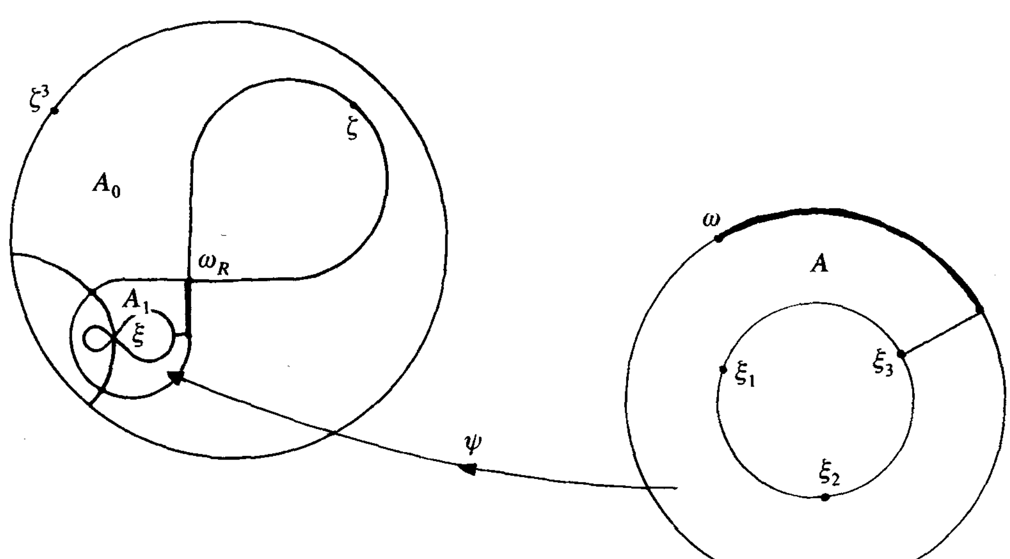


Fig. 2.6

2.8. The pattern bundle

The entire construction of $\mathcal{P}(\zeta)$ was functorial; $\mathcal{P}(\zeta)$ is defined up to unique isomorphism. As a result we can glue all these structures together to make a "bundle" over $\mathbf{C}-\bar{D}$.

PROPOSITION 2.12. *For any $N\in\mathbf{Z}$ there exists a unique analytic structure on the disjoint union*

$$U_N=\bigcup_{\zeta\in\mathbf{C}-\bar{D}}\;\bigcup_{R\in\mathcal{P}_N(\zeta)} R$$

making this set into a 2-dimensional complex manifold, such that:

(a) *The union $\omega=\bigcup\omega_R$ is for each $N\geq 1$ a smooth analytic curve;*

(b) *The map $\pi_N\colon U_N\to U_{N-1}$ induced by the ramified triple covering maps*

$$\pi_R\colon R\to s_N(R)$$

is a triple ramified covering space, ramified along the curve ω;

(c) *The complex structure on U_0 is inherited from the identification*

$$U_0=\{(\zeta,z)\,|\,|z|>|\zeta|\}.$$

With this structure, the mapping $p_N\colon U_N\to\mathbf{C}-\bar{D}$ which associates ζ to a point x of $R\in\mathcal{P}_N(\zeta)$ is an analytic submersion.

Proof. One must show that the construction can be done with parameters. This presents no difficulties and is left to the reader. Q.E.D. for Proposition 2.12

We will think of $p_N\colon U_N\to\mathbf{C}-\bar{D}$ as a bundle of Riemann surfaces where each fiber consists of $(1+3^{N-1})/2$ Riemann surfaces. The bundle is *not* analytically locally trivial: Propositions 2.8 and 2.9 say exactly when two fibers are isomorphic. It is topologically locally trivial, and we will study its monodromy in Chapter 6.

The reader will observe that all the structure of patterns made its way into the construction above, except that no analogue of the mappings i_R is mentioned. Recall that the mapping $i_R\colon s_N(R)\to R$ is labelled by its *range rather than its domain*. This is necessary, because for $N\geq 2$ a point $(\zeta,x)\in U_{N-1}$ belongs to a pattern $R\in\mathcal{P}_{N-1}(\zeta)$ which can be extended to a pattern of depth N in several ways, all of which are of interest. However, there is a mapping defined on a subset of U_{N-1}, which is a restriction of some $i_{R'}$, for all the R' with $s_N(R')=R$.

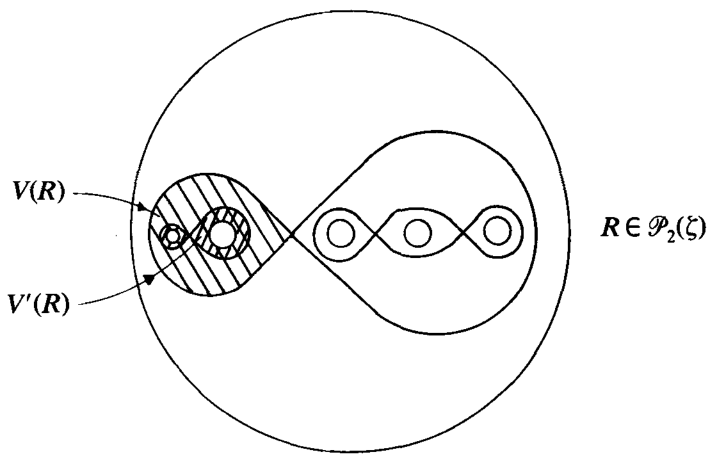


Fig. 2.7

Each R contains the sub-Riemann surface $V(R)$ consisting of the annulus $B_{N-1}(R)$ and all points inside it, and $V'(R)=V(R)-\overline{B_{N-1}(R)}$, which is a disjoint union of annuli. See Figure 2.7.

Define for $N\geq 0$ the corresponding open subsets $V'_N\subset V_N\subset U_N$ of the pattern bundle of depth N by

$$V_N=\bigcup_{\zeta\in\mathbf{C}-\bar{D}}\ \bigcup_{R\in\mathcal{P}_N(\zeta)} V(R),$$

and

$$V'_N=\bigcup_{\zeta\in\mathbf{C}-\bar{D}}\ \bigcup_{R\in\mathcal{P}_N(\zeta)} V'(R).$$

By the construction in Section 2.2, if $(\zeta,x)\in V'(R)\subset V'_{N-1}$ there exists a unique $R'\in\mathcal{P}_N(\zeta)$ such that $s_N(R')=R$ and $i_{R'}(x)\in B_{N-1}(R')\subset V(R')$. This defines for $N\geq 1$ a mapping

$$i_N\colon V'_{N-1}\to V_N.$$

PROPOSITION 2.13. *The mapping $i_N\colon V'_{N-1}\to V_N$ is injective and analytic.*

Proof. The map i_N is injective since each $i_{R'}$ is injective. The proof of analyticity must go by induction on N since the analytic structure on U_N is defined by induction. The space V'_0 is by definition the set $\{(\zeta,z)\mid |\zeta|<|z|<|\zeta|^3\}$, and the composition of $\pi_1\circ i_1$ is precisely the map $(\zeta, z) \to (\zeta, z^3)$. Since π_1 is a ramified covering space and i_1 is continuous, this shows that i_1 is analytic.

From the definition of i_N and from $i_{s_N(R)} \circ \pi_{s_N(R)} = \pi_R \circ i_R$ we get for each $N \geq 2$ that the following diagram commutes

$$\begin{array}{ccc} V'_{N-1} & \xrightarrow{i_N} & V_N \\ \downarrow{\scriptstyle \pi_{N-1}} & & \downarrow{\scriptstyle \pi_N} \\ V'_{N-2} & \xrightarrow{i_{N-1}} & V_{N-1} \end{array}$$

Since π_{N-1} and π_N are analytic covering maps the result follows by induction.

Q.E.D. for Proposition 2.13

2.9. The quotient pattern bundle

The statements about isomorphisms of patterns in Section 2 have immediate generalizations to the bundles under consideration here.

PROPOSITION 2.14. (a) *The mapping $\tau_N: U_N \to U_N$ induced by $\tau_R: R \to \tau(R)$ for each $R \subset U_N$ define for each N an analytic involution covering the involution $z \mapsto -z$ of $\mathbf{C} - \bar{D}$.*

(b) *The group of bundle-automorphisms of U_N for $N > 1$ is* $\{\mathrm{id}, \tau_N\}$.

Proof. The mapping $\tau_0: U_0 \to U_0$ is simply $(\zeta, z) \mapsto (-\zeta, -z)$, hence analytic. Now the result follows by induction, as usual. Q.E.D. for Proposition 2.14

In Chapter 6 we will consider the quotient bundle $\tilde{p}_N: \tilde{U}_N \to \mathbf{C} - \bar{D}$, where $\tilde{U}_N = U_N/\tau_N$, and $\tilde{p}_N([x]) = (p_N(x))^2$ so the fiber above ζ in the bundle $p_N: U_N \to \mathbf{C} - D$ is naturally isomorphic to the fiber above ζ^2 in the quotient bundle $\tilde{p}_N: \tilde{U}_N \to \mathbf{C} - \bar{D}$.

Chapter 3. Patterns and polynomials

Throughout this chapter, P will be a monic cubic polynomial, with critical points ω_1, ω_2 satisfying $h_P(\omega_2) < h_P(\omega_1)$. Set

$$U_P(z_0) = \{z \in \mathbf{C} \mid h_P(z) > h_P(z_0)\}.$$

In this chapter we will show that subsets $U_P(z_0)$ are isomorphic to subsets of patterns when $h_P(\omega_2) < h_P(z_0)$.

More precisely, suppose $h_P(\omega_1)/3^{N+1} \leq h_P(\omega_2) < h_P(\omega_1)/3^N$. Then we will show that there exists a $\zeta \in \mathbf{C} - \bar{D}$, an abstract Riemann surface $R \in \mathcal{P}_{N+1}(\zeta)$ and an extension of φ_P to an analytic isomorphism

$$\bar{\varphi}_P \colon U_P(\omega_2) \to R(P),$$

where $R(P)$ is an appropriate subset of R.

If ω_2 does not escape to ∞, we will show that $\mathbf{C} - K_P$ is isomorphic to an appropriate ζ-pattern of infinite depth.

The precise statements are contained inTheorem 3.1 and Corollary 3.5.

3.1. Polynomials realizing patterns

If

$$\frac{h_P(\omega_1)}{3^{N+1}} \leq h_P(\omega_2) < \frac{h_P(\omega_1)}{3^N}$$

P will be said to have *depth* N. The polynomial P will have *critical depth* if the left hand inequality is an equality; if $h_P(\omega_2) = 0$, i.e. if $\omega_2 \in K_P$, we say that P has *infinite depth*. Theorem 3.1 and Remark 3.2 justify this terminology.

THEOREM 3.1. *Suppose* depth$(P) = N \geq 0$. *Let* ω_1' *be the co-critical inverse image of* $P(\omega_1)$, *and set*

$$\lim_{z \to \omega_1'} \varphi_P(z) = \zeta.$$

Then there exists a unique ζ-pattern $R \in \mathcal{P}_{N+1}(\zeta)$ and a unique analytic isomorphism $\bar{\varphi}_P \colon U_P(\omega_2) \to R(P)$ where $R(P) \subset R$ is defined by

$$R(P) = \{z \in R \mid h_R(z) > h_P(\omega_2)\},$$

such that the diagram

$$\begin{array}{ccc} U_P(\omega_1) & \subset & U_P(\omega_2) \\ \Big\downarrow{\scriptstyle \varphi_P} & & \Big\downarrow{\scriptstyle \bar{\varphi}_P} \\ R_0 & \subset & R(P) \end{array}$$

commutes, where $R_0 = \{z \in R \mid h_R(z) > \log|\zeta|\}$.

Remark 3.2. Theorem 3.1 shows that there is a pattern R of depth $N+1$ associated to every polynomial of depth N. We have the following inclusions $R_N \subset R(P) = \bar{\varphi}_P(U_P) \subset R$ where the right inclusion is an equality only if the polynomial has critical depth N. Hence we say that the polynomial P is a *realization* of the ζ-pattern R to depth N. The theorem does not assert that every pattern of depth N can be realized by some polynomial of depth N. This is true, and will be shown in Theorem 8.2. To show this, we will need to evaluate $\bar{\varphi}_P$ at critical values; the Corollary 3.3 below will allow this.

For $n \le N$ we define *the critical annulus* $C_n(P)$ of P at level n to be $C_n(P) = \bar{\varphi}_P^{-1}(C_n(R))$, i.e. the connected component of the potential strip of level n surrounding the critical point ω_2, and we define *the critical value annulus* $B_{n-1}(P)$ of P at level $n-1$ to be $B_{n-1}(P) = \bar{\varphi}_P^{-1}(B_{n-1}(R))$, i.e. the connected component of the potential strip of level $n-1$ surrounding the critical value $P(\omega_2)$.

Proof of Theorem 3.1. Set

$$U_n = P^{-n}(U_P(\omega_1)) = \left\{ z \in \mathbf{C} \;\middle|\; h_P(z) > \frac{h_P(\omega_1)}{3^n} \right\}$$

the proof must go by induction on n since that is the way the patterns are constructed.

Set $M=N$ unless P is of critical depth then set $M=N+1$. We will construct for $n \le N+1$ a sequence of patterns $R_n \in \mathcal{P}_n(\zeta)$ with $s_n(R_n) = R_{n-1}$ and for $n \le M$ a sequence of analytic isomorphisms $\varphi_n \colon U_n \to R_n$ such that the diagrams

$$(*) \qquad \begin{array}{ccc} U_n & \xrightarrow{P} & U_{n-1} \\ \downarrow{\scriptstyle \varphi_n} & & \downarrow{\scriptstyle \varphi_{n-1}} \\ R_n & \xrightarrow{\pi_{R_n}} & R_{n-1} \end{array}$$

commute. To start the induction set $\varphi_0 = \varphi_P$ and define ζ by $\zeta = \lim_{z \to \omega_1'} \varphi_P(z)$.

If $M>0$, make the following inductive hypotheses: for all $0<m\le M$

— For all $n<m$ there exist unique $R_n \in \mathcal{P}_n(\zeta)$ and isomorphisms $\varphi_n \colon U_n \to R_n$ which make the diagrams $(*)$ above commute.

If $\eta \in \mathbf{R}$ satisfies $1<\eta<3$, set $U_{n,\eta} = \{z \in \mathbf{C} \mid h_P(z) > \eta \log|\zeta|/3^n\}$; we have the inclusions $U_{n-1} \subset U_{n,\eta} \subset U_n$.

— The restriction $\varphi_{n,\eta}$ of φ_n to $U_{n,\eta}$ extends as a homeomorphism to $\bar{\varphi}_{n,\eta} \colon \mathbf{C} \to \bar{R}_n$ for all $n<m$.

We need to introduce $\eta>1$ for the following reason: each component of $\mathbf{C} - U_{n,n}$ is homeomorphic to a closed disc, and these correspond bijectively to the components of $\bar{R}_n - R_n$. This is not true if $\eta=1$; the components of the boundary of U_n are graphs with double points. So in particular, the second inductive hypothesis is not true if $\eta=1$.

Since $m \leq N$, the point $\bar{\varphi}_{m-1,\eta}(P(\omega_2)) \in \bar{R}_{m-1} - R_{m-1,\eta}$ where

$$R_{m-1,\eta} = \bar{\varphi}_{m-1,\eta}(U_{n,\eta});$$

the mapping $\bar{\varphi}_{m-1,\eta}$ can be modified so that

$$x = \bar{\varphi}_{m-1,\eta}(P(\omega_2)) \in \bar{R}_{m-1} - \mathrm{clos}(R_{m-1})$$

without modifying it on $U_{n,\eta}$. Then x is separated from ∞ by a unique component A of $R_{m-1} - \mathrm{clos}(i_{R_{m-1}}(R_{m-2}))$, which is an annulus nested within $B_{m-2}(R_{m-1})$. This annulus (and the choice of x) selects the element $R_m = R_A \in \mathcal{P}_m(\zeta)$, and by Lemma 2.2 there exists a unique homeomorphism

$$\psi_\eta : \mathbf{C} \to \bar{R}_m$$

such that the diagram

$$(**) \qquad \begin{tikzcd} \mathbf{C} \arrow[r, "P"] \arrow[d, "\psi_\eta"'] & \mathbf{C} \arrow[d, "\bar{\varphi}_{m-1,\eta}"] \\ \bar{R}_m \arrow[r, "\pi_{\bar{R}_m}"] & \bar{R}_{m-1} \end{tikzcd}$$

commutes. If $1 < \eta_1 < \eta_2$ then $\psi_{\eta_1} = \psi_{\eta_2}$ on U_{m,η_2}. Therefore $\varphi_m = \lim_{\eta \to 1} \psi_\eta$ is well defined and makes the diagram (*) commute. Moreover, the maps ψ_η are extensions verifying the second inductive hypothesis.

If P is of critical depth set $R = R_{N+1}$ and $\bar{\varphi}_P = \varphi_{N+1}$. If P is not of critical depth then choose $\eta = 3^{N+1} h_P(\omega_2)/h_P(\omega_1)$ so that $U_{N+1,\eta} = U_P(\omega_2)$. Construct R_{N+1} and $\psi_\eta : \mathbf{C} \to \bar{R}_{N+1}$ as above and set $R = R_{N+1}$ and $\bar{\varphi}_P = \psi_\eta|_{U_P(\omega_2)}$. The sets $R_0 = s_1 \circ \ldots \circ s_{N+1}(R)$ and $\{z \in R \mid h_R(z) > \log|\zeta|\}$ are canonically isomorphic so we can identify them.

If there were another pattern R' satisfying the requirements of the theorem, then the isomorphism $\bar{\varphi}'_P$ would induce isomorphisms φ'_m at all levels. By hypothesis, $\varphi'_0 = \varphi_0 = \varphi_P$, and now the equality $\varphi'_m = \varphi_m$ follows by induction from the uniqueness in Lemma 2.2. Q.E.D. for Theorem 3.1

COROLLARY 3.3. *Suppose* depth$(P)=N\geq 0$. *Then the critical value* $P(\omega_2)$ *lies in* $U_P(\omega_2)$ *hence is in the domain of* $\bar{\varphi}_P$.

Remark 3.4. Theorem 3.1 and Corollary 3.3 can be extended to the case where $h_P(\omega_1)=h_P(\omega_2)$ and $\omega_1\neq w_2$ (i.e. critical depth -1). The mapping φ_P can then be extended to a neighborhood of the co-critical point ω_1' and it is still true that $\zeta\in\mathbf{C}-\bar{D}$ is uniquely determined. However, ζ depends not just on P but also on the order of the critical points. Hence Theorem 3.1 and Corollary 3.3 do not apply to the case where $\omega_1=\omega_2$, there is then no co-critical point, so ζ is not defined.

COROLLARY 3.5. *Suppose* depth$(P)=\infty$. *Let* ω_1' *be the co-critical inverse image of* $P(\omega_1)$, *and set* $\zeta=\lim_{z\to\omega_1'}\varphi_P(z)$. *Then there exists a unique* ζ*-pattern* R_∞ *of infinite depth and a unique analytic isomorphism* $\bar{\varphi}_P\colon\mathbf{C}-K_P\to R_\infty$ *which extends* φ_P. *The isomorphism* $\bar{\varphi}_P$ *conjugates* P *to* π_{R_∞}.

The proof is essentially identical with the proof of Theorem 3.1.

Remark 3.6. Corollary 3.5 shows that there is a pattern R_∞ of infinite depth associated to every polynomial of infinite depth. We say that the polynomial P is a *realization* of the ζ-pattern R_∞.

The corollary does not assert that every pattern of infinite depth can be realized by some polynomial of infinite depth, but this is also true, and will be shown in Theorem 9.1.

3.2. Introducing parameters in Theorem 3.1

Let Λ be a complex manifold, and let $(P_\lambda)_{\lambda\in\Lambda}$ be an analytic family of monic cubic polynomials parametrized by Λ, and with distinct critical points $\omega_1(\lambda)$, $\omega_2(\lambda)$ satisfying

$$h_\lambda(\omega_2(\lambda))<\frac{h_\lambda(\omega_1(\lambda))}{3^N},$$

i.e. all P_λ have depth $\geq N$. Let $\omega_1'(\lambda)$ be the co-critical inverse image of $P_\lambda(\omega_1(\lambda))$. Set

$$\zeta(\lambda)=\lim_{z\to\omega_1'(\lambda)}\varphi_\lambda(z),$$

and

$$X_n=\left\{(\lambda,z)\,\middle|\,h_\lambda(z)>\sup\left(h_\lambda(\omega_2(\lambda)),\frac{h_\lambda(\omega_1(\lambda))}{3^n}\right)\right\}.$$

Define $\bar{\varphi}_\lambda = \bar{\varphi}_{P_\lambda}$ (from Theorem 3.1), and similarly $\varphi_{n,\lambda}$ the mapping φ_n constructed in the proof of Theorem 3.1, as applied to the polynomial P_λ.

The spaces U_N defined in Section 2.8 have the following universal property:

PROPOSITION 3.7. *The mapping* $\bar{\varphi}: X_{N+1} \to U_{N+1}$ *given by*

$$\bar{\varphi}(\lambda, z) = \begin{cases} \bar{\varphi}_\lambda(z) & \text{if } \operatorname{depth}(P_\lambda) = N \\ \varphi_{N+1,\lambda}(z) & \text{if } \operatorname{depth}(P_\lambda) > N \end{cases}$$

is analytic.

Let $p: X_{N+1} \to \Lambda$ *be the projection onto the first factor. Then the diagram*

$$\begin{tikzcd} X_{N+1} \arrow[r, "\bar{\varphi}"] \arrow[d, "p"'] & U_{N+1} \arrow[d, "p_{N+1}"] \\ \Lambda \arrow[r, "\lambda \mapsto \zeta(\lambda)"] & \mathbf{C} - \bar{D} \end{tikzcd}$$

commutes.

If $\operatorname{depth}(P_\lambda) \leq N+1$ *then the critical value* $P_\lambda(\omega_2(\lambda)) \in X_{N+1}$ *and* $\bar{\varphi}(P_\lambda(\omega_2(\lambda))) \in V_{N+1}$.

Proof. Define $\bar{\varphi}_n: X_n \to U_n$ by $\bar{\varphi}_n(\lambda, z) = \varphi_{n,\lambda}(z)$. The proof goes by induction on $n = 0, \ldots, N+1$, essentially putting parameters in the proof of Theorem 3.1. To start the induction, observe that $\bar{\varphi}_0(\lambda, z) = \varphi_\lambda(z)$, which depends analytically on (λ, z).

The diagrams

$$\begin{tikzcd} X_n \arrow[r, "(\lambda, z) \mapsto (\lambda, P_\lambda(z))"] \arrow[d, "\bar{\varphi}_n"'] & X_{n-1} \arrow[d, "\bar{\varphi}_{n-1}"] \\ U_n \arrow[r, "\pi_n"] & U_{n-1} \end{tikzcd}$$

commute, since they commute on fibers.

Since the diagram commutes and since $\bar{\varphi}_n: X_n \to U_n$ is a lifting of $\bar{\varphi}_{n-1}$ to a ramified covering space, it follows by induction that all $\bar{\varphi}_n$'s are analytic for $n \leq N+1$.

Q.E.D. for Proposition 3.7

Chapter 4. Ends of patterns

In Section 2.2 we defined the tree of Riemann surfaces $\mathcal{P}(\zeta)$ and in Section 2.6 the patterns in $\mathcal{P}_\infty(\zeta)$ of infinite depth. In this chapter we will investigate the ends of the Riemann surfaces of infinite depth.

In Section 3.1 we defined cubic polynomials of infinite depth and their associated patterns. In Chapter 5 we shall consider cubic polynomials of infinite depth. The components of the filled-in Julia set K_P of such a polynomial P are in 1–1 correspondence with the ends of the pattern R_∞ associated with P. We will discover in Chapter 5 exactly when a component in K_P is a point and when it is a continuum. The two cases correspond exactly to the two types of ends analyzed in this chapter: divergent ends and convergent ends respectively.

4.1. Patterns and their ends

In this section we will only be concerned with patterns of infinite depth, and will omit the words 'infinite depth' throughout. Recall that to each such pattern we can associate the Riemann surface

$$R_\infty = \varinjlim (R_n, i_{R_{n+1}})$$

and in this section we will drop the subscript ∞. The Propositions 2.8 and 2.9 say that R contains all the information about the pattern up to determination of $\pm\zeta$; most often we will identify a ζ-pattern and the corresponding Riemann surface.

We will be mainly concerned with the set of ends $E(R)$. Of course ∞ is an end of R, but we don't want to consider it, so we set

$$E(R) = \varprojlim \pi_0(\hat{R} - K),$$

where $\hat{R} = R \cup \{\infty\}$, and the projective limit is taken as usual over the directed system of compact subsets K of $\hat{R}$.

The topological set $E(R)$ is a Cantor set, and since the mapping $\pi_R: R \to R$ is proper, it induces a continuous mapping $E(\pi_R): E(R) \to E(R)$, which we will ordinarily still denote π_R.

The subsets $\operatorname{clos}(\hat{R}_n)$ form a compact exhaustion of $\hat{R}$, so this space of ends can also be described as

$$E(R) = \varprojlim \pi_0(R - R_n).$$

4.2. Nests and tableaux

Let R be a ζ-pattern. For any $x \in E(R)$, define the *nest* $N(x)$ of x to be the sequence $A_l(x)_{0 \leq l < \infty}$ of annuli of R, where $A_l(x)$ is the annulus at level l separating x from ∞. The nests of any two ends of R coincide up to some positive level and then differ from there on.

Set

$$\operatorname{mod} N(x) = \sum_{l=0}^{\infty} \operatorname{mod}(A_l(x)).$$

We will call the end $x \in E(R)$ *convergent* if

$$\operatorname{mod} N(x) < \infty$$

divergent otherwise. If we realize R as an open subset of $\mathbf{C}$, with ∞ at ∞, through a polynomial P, then each end of R will correspond to a component of $\mathbf{C}-R$, and conversely. We will see in Section 5.3 that if an end is divergent, then the corresponding component must be reduced to a point. As we will see in Theorem 5.2 it is also true for a cubic polynomial P that if an end is convergent then the corresponding component is not reduced to a point.

One end of any ζ-pattern R is distinguished: the critical end $c \in E(R)$, whose nest, the *critical nest*, $N(c)$ consists of the *critical annuli* $C_0, C_1, \ldots$, which map to their images under π_R by double covers.

In order to estimate $\operatorname{mod} N(x)$ for an end $x \in E(R)$, we will define its *tableau* $T(x)$; this is the two dimensional array of annuli $A_{l,k}(x)$, where $A_{l,0}(x) = A_l(x)$ is the annulus of $N(x)$ at level l, and $A_{l,k}(x) = \pi_R^{\circ k}(A_{l+k,0}(x))$. Another way of saying this is that the kth column is the nest $N(\pi_R^{\circ k}(x))$ of $\pi_R^{\circ k}(x)$ so that $A_{l,k}(x) = A_{l,0}(\pi_R^{\circ k}(x))$. See Figure 4.1.

$N(x)$	$N(\pi_R(x))$		$N(\pi_R^{\circ k}(x))$	
$A_{0,0}(x)$	$A_{0,1}(x)$	$\cdots$	$A_{0,k}(x)$	$\cdots$
$A_{1,0}(x)$	$A_{1,1}(x)$	$\cdots$	$A_{1,k}(x)$	$\cdots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$
$A_{l,0}(x)$	$A_{l,1}(x)$	$\cdots$	$A_{l,k}(x)$	$\cdots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$

Fig. 4.1. The tableau $T(x)$.

Since C_0 is the only annulus at level 0 we have $A_{0,k}(x)=C_0$ for all k and all x.

The map $\pi_R: A_{l,k}(x)\to A_{l-1,k+1}(x)$ is a double cover if $A_{l,k}(x)=C_l$ and a simple cover otherwise. The *critical tableau* $T(c)$ is the tableau of the critical end.

Of primary interest in a tableau are the *critical positions*, i.e. the pairs (l, k) such that $A_{l,k}(x)=C_l$. For instance, the critical end is periodic of period dividing k if and only if the kth column of the critical tableau is entirely critical, and an end x is a preimage of the critical end if and only if some column of $T(x)$ is entirely critical.

All moduli of annuli can be computed from the critical positions. The modulus of $A_{l,k}(x)$ satisfies

$$\operatorname{mod}(A_{l,k}(x)) = \frac{1}{2^n}\operatorname{mod}(C_0).$$

where n is the number of critical positions (i,j) along the diagonal $i+j=l+k$ and with $0<i\leq l$.

4.3. Properties of tableaux

PROPOSITION 4.1. *All the tableaux for a given ζ-pattern satisfy the following* 3 *tableau rules:*

(Ta) *The critical positions form unbroken vertical lines stretching out from the top of a tableau, i.e. if $A_{l,k}(x)=C_l$ then $A_{j,k}(x)=C_j$ for all $0\leq j\leq l$.*

(Tb) *If $A_{l,k}(x)=C_l$ then $A_{i,k+j}(x)=A_{i,j}(c)$ for $i+j\leq l$;*

(Tc) *If for the critical tableau $T(c)$ and for some $l\geq n>0$*

— $A_{l+1-n,n}(c)=C_{l+1-n}$ *and* $A_{l-i,i}(c)\neq C_{l-i}$ *for* $0<i<n$,

and if for any tableau $T(x)$ and for some $m>0$

— $A_{l,m}(x)=C_l$ *and* $A_{l+1,m}(x)\neq C_{l+1}$,

then $A_{l+1-n,m+n}(x)\neq C_{l+1-n}$.

Proof. Properties (Ta) and (Tb) are fairly clear:

(a) The vertical columns are nests, which always agree up to some point with the critical nest and then separate.

(b) Down to level l the kth column in the tableau $T(x)$ is the same as the 0-column of the critical tableau. A complete triangle is copied from the critical tableau to the tableau $T(x)$, since

$$A_{i,k+j}(x) = \pi_R^{\circ j}(A_{i+j,k}(x)) = \pi_R^{\circ j}(C_{i+j}) = \pi_R^{\circ j}(A_{i+j,0}(c)) = A_{i,j}(c).$$

Property (Tc) is more technical; it is seen as follows:

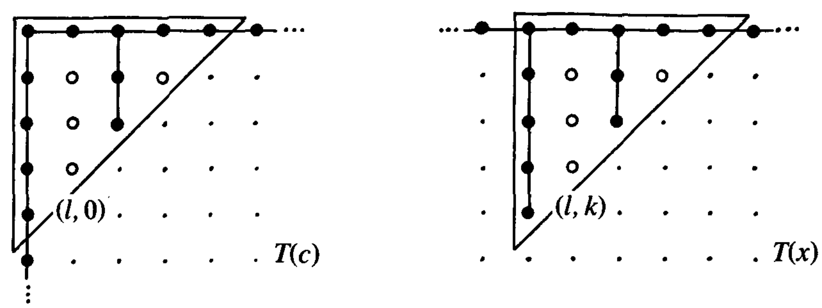


Fig. 4.2. Rule (Tb).

(c) The first part of the hypothesis says that the image of C_l under $\pi_R^{\circ n}$ is C_{l-n}, that $\pi_R^{\circ n}$ restricted to C_l and all points inside it is of degree 2 and that the inverse image of C_{l+1-n} under this restriction has C_{l+1} as one component. Now since the degree of the restriction is 2, the C_{l+1} is the entire inverse image, so there cannot be in addition a non-critical annulus in the inverse image. Q.E.D. for Proposition 4.1

Most often when discussing a tableau for a given ζ-pattern we will disregard all information except the critical positions, and think of it as simply the *grid* $\mathbf{N}^2$ with critical positions *marked*. The rules (Ta), (Tb), (Tc) above can be formulated in terms of marked grids. Of particular interest is the *critical marked grid* with all positions in the 0-column marked. We will mark a critical position by • and a non-critical position by ∘. The tableau rules (Tb) and (Tc) are illustrated in Figures 4.2 and 4.3 respectively.

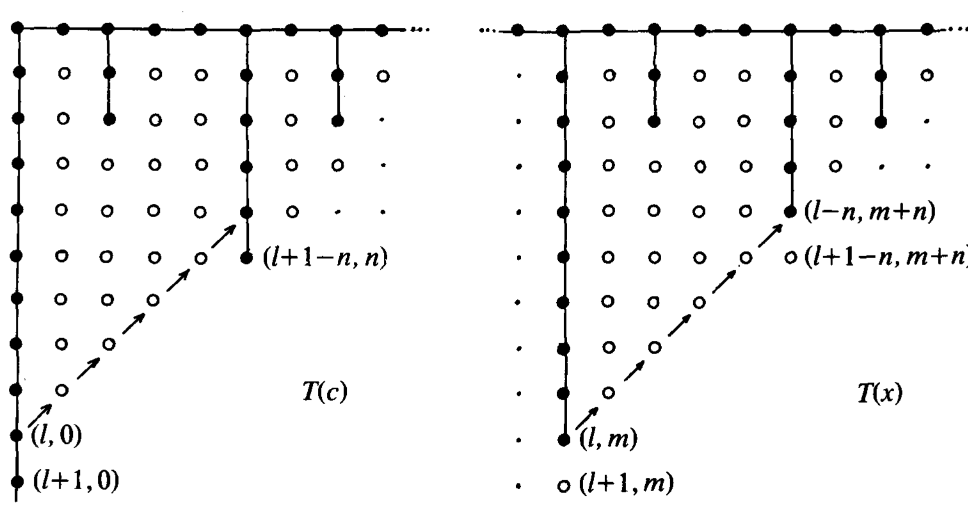


Fig. 4.3. Rule (Tc).

4.4. The realizability of tableaux

Properties (Ta), (Tb), (Tc) of tableaux capture much of what the dynamics actually allows, as the following proposition shows.

THEOREM 4.1. (a) *Any critical marked grid G satisfying the rules* (Ta), (Tb), (Tc) *can be realized for any $\zeta\in\mathbf{C}-\bar{D}$ as the critical tableau for some pattern $R\in\mathcal{P}_\infty(\zeta)$; in general the ζ-pattern is not uniquely determined from the marked grid.*

(b) *Given a pattern $R\in\mathcal{P}_\infty(\zeta)$, any marked grid satisfying* (Ta), (Tb), (Tc) *with respect to the critical tableau of R can be realized by an end of R. In general the end is not uniquely determined from the marked grid.*

Proof. (a) The proof goes by induction. A pattern $R_n\in\mathcal{P}_n(\zeta)$ has a *restricted critical tableau* $\{A_{i,j}(c)\}_{i+j\leq n}$; suppose we have found one whose critical positions (i,j) coincide with our critical marked grid G for $i+j\leq n$. We will show that there is at least one choice of $R_{n+1}\in\mathcal{P}_{n+1}(\zeta)$ with $s_{n+1}(R_{n+1})=R_n$ whose restricted critical tableau has critical positions agreeing with those of G in the range $i+j\leq n+1$. We shall show this by choosing annuli $A_{i,n+1-i}(c)$ of the pattern R_n for $i=1,\dots,n$; the choice of $A_{n,1}(c)$ will tell us which annulus the critical value annulus at level n should be and hence determine R_{n+1}.

First, let $(k,n+1-k)$ be the critical position on the line $i+j=n+1$ with the largest k, $0\leq k\leq n$; in the position $(i,n+1-i)$ with $0\leq i\leq k$ recopy the annuli in positions $(i,k-i)$. By property (Tb) these positions will be critical if and only if the corresponding positions of G are critical, moreover the choices are compatible with π_{R_n} and nesting.

Now successively fill the positions $(j+1,n-j)$ for $j=k,k+1,\dots,n-1$. This can be done without ambiguity so long as the position $(j,n-j)$ is not critical. In that case $\pi_{R_n}\colon A_{j,n-j}(c)\to A_{j-1,n-j+1}(c)$ is of degree 1 so there is a unique annulus nested in $A_{j,n-j}(c)$ and mapped onto $A_{j,n-j+1}(c)$ by π_{R_n}. If the position $(j,n-j)$ is critical, the analysis is involved; we will call such cases *ambiguous cases*.

Suppose $(j+1,n-j)$ is an ambiguous case and let $m>0$ be that integer such that $(j-m,m)$ is critical and all the positions $(j-i,i)$ for $0<i<m$ are non-critical, see Figure 4.4. By property (Tb) the position $(j-m,n-j+m)$ is critical. The analysis divides into 2 cases according to whether the position $(j-m+1,m)$ is critical or not.

If the position $(j-m+1,m)$ is critical then by property (Tc) the position $(j-m+1,n-j+m)$ is not critical. The annulus $A_{j,n-j+1}(c)$ is different from $A_{j,1}(c)$, since $\pi_{R_n}^{\circ(m-1)}(A_{j,1}(c))=C_{j-m}$ and $\pi_{R_n}^{\circ(m-1)}(A_{j,n-j+1}(c))\neq C_{j-m}$, so it has two (non-critical) inverse images in $A_{j,n-j}(c)=C_j$. Hence the annulus $A_{j+1,n-j}(c)$ can be chosen in two ways, but as we shall see we may have to change our choice at the next ambiguous case.

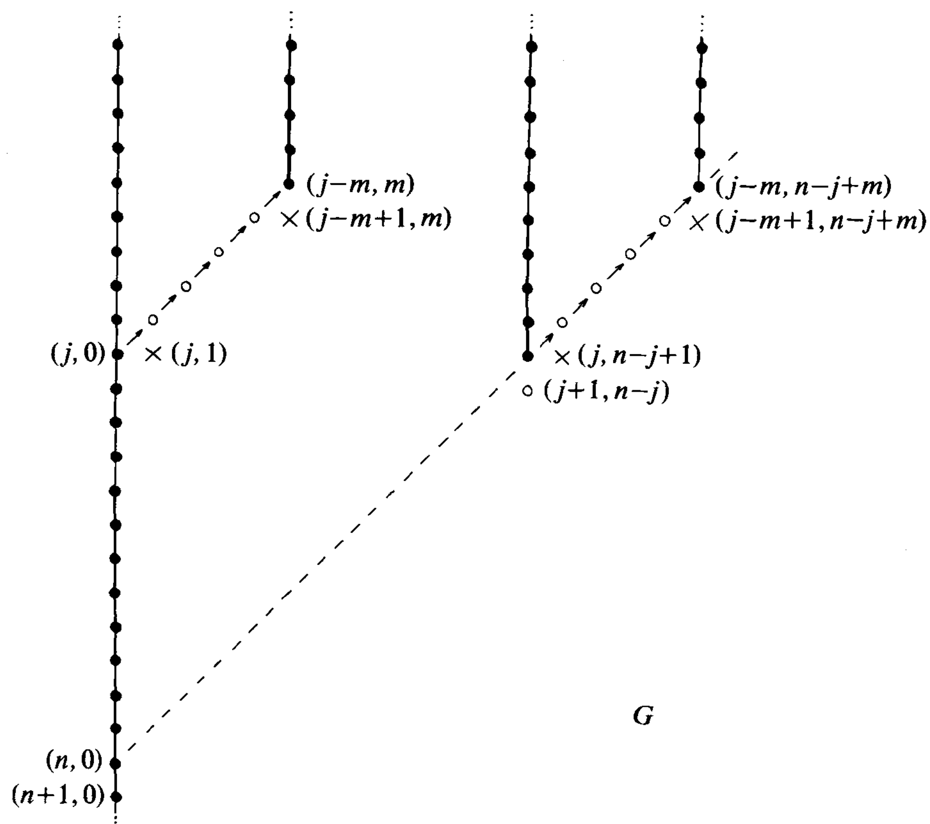


Fig. 4.4

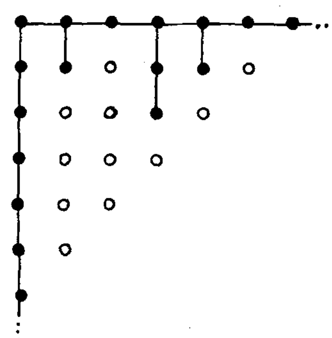
If the position $(j-m+1, m)$ is not critical then there are 2 cases according to whether $(j-m+1, n-j+m)$ is critical or not.

If $(j-m+1, n-j+m)$ is critical and $(j-m+1, m)$ is not, then the annulus $A_{j,n-j+1}(c)$ is different from $A_{j,1}(c)$, since

$$\pi_{R_n}^{\circ(m-1)}(A_{j,1}(c)) \neq C_{j-m} \quad \text{and} \quad \pi_{R_n}^{\circ(m-1)}(A_{j,n-j+1}(c)) = C_{j-m},$$

so it has two (non-critical) inverse images in $A_{j,n-j}(c)=C_j$. As above the annulus $A_{j+1,n-j}(c)$ can be chosen in two ways.

If neither $(j-m+1, n-j+m)$ nor $(j-m+1, m)$ is critical, then change if necessary the choice of the annulus $A_{j-m+1,n-j+m}(c)$ to make it different from $A_{j-m+1,m}(c)$. This will further change the $A_{j-m+1+i,n-j+m-i}(c)$ for $0<i<m$, in an unambiguous way. Then the annulus $A_{j,n-j+1}(c)$ is different from $A_{j,1}(c)$, since



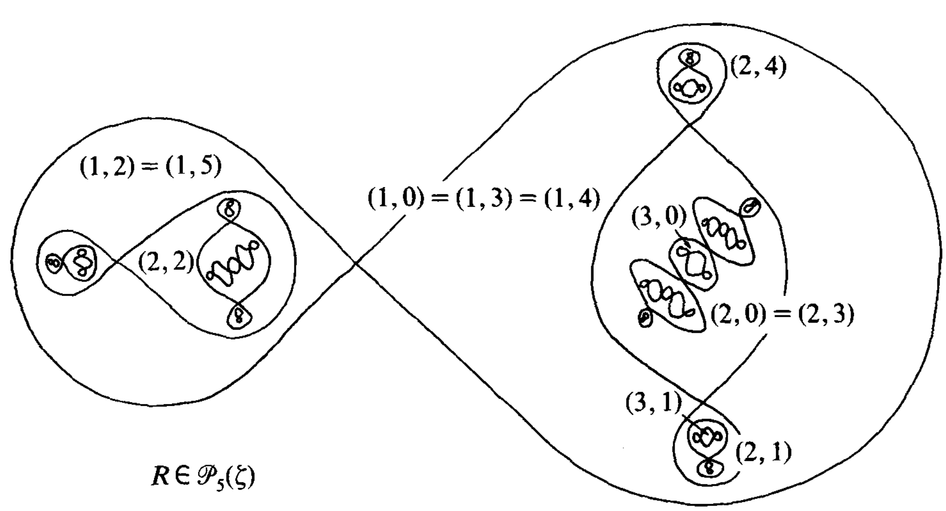
Restricted critical marked grid; $i+j \leq 6$.

Fig. 4.5

$$\pi_{R_n}^{\circ(m-1)}(A_{j,1}(c)) = A_{j-m+1,m}(c) \neq A_{j-m+1,n-j+m}(c) = \pi_{R_n}^{\circ(m-1)}(A_{j,n-j+1}(c)).$$

Hence as above the annulus $A_{j+1,n-j}(c)$ can be chosen in two ways. Moreover, we will never have to change $A_{j-m+1,n-j+m}(c)$ again.

(b) The proof is almost identical with the proof of (a). By induction on n we choose annuli $\{A_{i,j}\}_{i+j\leq n}$ of R_∞. Suppose this has been done for the positions $\{(i,j)\}_{i+j\leq n}$. Let



One realization of the restricted critical grid in Figure 4.5.

Fig. 4.6

$(k, n+1-k)$ be the critical position on the line $i+j=n+1$ with the largest k, $0 \leqslant k \leqslant n+1$. Then we have to choose $A_{i,n+1-i}=A_{i,k-i}(c)$, $0 \leqslant i \leqslant k$. If $k<n+1$ we fill the positions $(j+1, n-j)$ successively for $j=k, \ldots, n$. This can be done without ambiguity so long as the position $(j, n-j)$ is not critical. If $(j+1, n-j)$ is an ambiguous case, i.e. if $(j, n-j)$ is critical, let $m>0$ be that integer such that $A_{j-m,m}(c)=C_{j-m}$ and $A_{j-i,i}(c) \neq C_{j-i}$ for $0<i<m$. Similar to the proof of (a) we can always arrange that $A_{j,n-j+1} \neq A_{j,1}(c)$, but we may have to change the previous choice $A_{j-m+1,n-j+m}$. Q.E.D. for Theorem 4.2

Example. Figure 4.5 shows a restricted critical grid with marked positions for $i+j \leqslant 6$ and Figure 4.6 shows a choice of annuli for $i+j \leqslant 5$. For each ζ the grid can be realized by two ζ-patterns in $\mathcal{P}_5(\zeta)$, since the annulus $A_{2,1}(c)$ can be chosen in two ways. If $R \in \mathcal{P}_5(\zeta)$ is as chosen in Figure 4.6 and if we choose $A_{2,4}(c)=A_{2,1}(c)$ we are in a situation where we have to change that choice. Therefore we are forced to choose $A_{2,4}(c)$ as shown and there is a unique extension from $R \in \mathcal{P}_5(\zeta)$ to a ζ-pattern in $\mathcal{P}_6(\zeta)$.

4.5. Moduli of nests

We shall now estimate $\operatorname{mod} N(x)$ for an end $x \in E(R)$ using its marked grid. Theorem 4.3 below gives a complete answer to the problem. The proof appears after Lemmas 4.4–4.6, which it requires.

THEOREM 4.3. (a) *The critical end $c \in E(R)$ is divergent if and only if it is not periodic.*

(b) *If the critical end $c \in E(R)$ is divergent then any end $x \in E(R)$ is divergent.*

(c) *If the critical end $c \in E(R)$ is convergent then an end $x \in E(R)$ is convergent if and only if it is precritical, i.e. if there exists an n with $\pi_R^{\circ n}(x)=c$.*

LEMMA 4.4. *If $x \in E(R)$ is a convergent end, then to the right of any position in its tableau there is a critical position.*

Proof. The modulus of $A_{l,0}(x)$ is expressed by

$$\operatorname{mod}(A_{l,0}(x)) = \frac{1}{2^n} \operatorname{mod}(C_0),$$

where n is the number of critical positions along the diagonal $i+j=l$ with $0<i\leqslant l$. If the horizontal line beginning at the position (m, k) has no critical positions to the right of (m, k), then by rule (Ta) the position (i, j) is non-critical whenever $i \geqslant m$ and $j>k$. Hence $n \leqslant m+k$. For any l

$$\text{mod}(A_{l,0}(x)) \geqslant \frac{1}{2^{m+k}} \text{mod}(C_0),$$

and clearly the end is divergent. Q.E.D. for Lemma 4.4

We will call the critical end *recurrent* if its tableau satisfies the conclusion of Lemma 4.5, i.e. if to the right of every position there is a critical position.

Define a position $(l,k) \neq (0,0)$ of the *critical tableau* to be an *originator* if and only if either

$l=0$ and there does not exist an i, $0<i<k$ with $(k-i,i)$ critical

or

$l>0$ and

(i) (l,k) is a critical position;

(ii) there exists i with $0<i<k$ such that (l,i) is critical;

(iii) all $(l+i,k-i)$ for $0<i<k$ are non-critical.

Clearly $(0,1)$ is an originator. We shall mark an originator by ◉.

LEMMA 4.5. *If the critical end $c \in E(R)$ is recurrent, then*

$$\text{mod}(N(c)) = \sum_{j=0}^{\infty} \text{mod}(C_j) = \text{mod}(C_0) + \sum_{(l,k)\ \text{originator}} \text{mod}(A_{l,k}(c)).$$

Proof. For each originator consider the subsequence of the critical nest constructed as follows: go down diagonally to the 0th column, then horizontally until you reach a critical annulus which exists since c is recurrent, then down diagonally until you reach the 0th column, etc. (see Figure 4.7). The moduli of the annuli on the 0th column you meet this way form a geometric series with ratio 1/2, which starts with 1/2 the modulus of the originator, thus the series sums to the modulus of the originator. Clearly each annulus except C_0 of the critical nest is reached exactly once in this way.

Q.E.D. for Lemma 4.5

The next result is really the crucial point.

LEMMA 4.6. *If the critical end $c \in E(R)$ is recurrent and non-periodic, then there is an originating position in every row.*

Proof. By induction, suppose that the annulus at the position (l,k) is critical and originating, and let $(l+1,k')$ be the first critical annulus on or to the right of $(l+1,k)$. If in the $l+1$ row in position $1,\ldots,k-1$ there are any critical annuli, then $(l+1,k')$ is

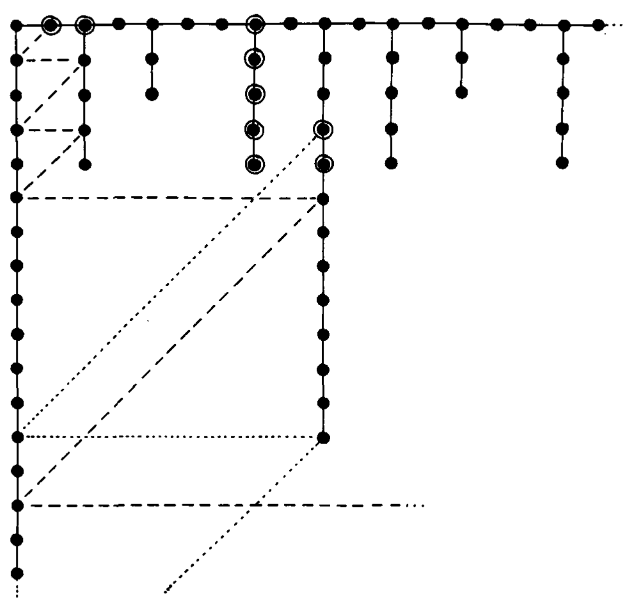


Fig. 4.7. The critical tableau.

originating. So assume there are none. The kth column is not entirely critical, since the critical end is assumed non-periodic, so continue the line of critical positions through $(l+1, k')$ to its end (l', k') and consider the diagonal just beyond its tip, i.e. the line of positions (i,j) satisfying $i+j=l'+k'+1$, see Figure 4.8.

Claim. There are no critical positions on this line with first coordinate greater than $l+1$.

If there were such a critical annulus, then by property (Tb) of tableaux it would have to appear in some position

$$(l'-(m-1)k'+1, mk'), \quad \text{for some} \quad 1<m<(l'-l+k')/k';$$

let m_0 be the smallest m for which this occurs. Then

— the positions $(l'-(m_0-1)k', k')$, $(l'-(m_0-1)k'+1, k')$ are critical and none of the positions $(l'-(m_0-2)k'-i, i)$ are critical for $0<i<k'$,

— the position $(l'-(m_0-2)k', (m_0-1)k')$ is critical and the position $(l'-(m_0-2)k'+1, (m_0-1)k')$ is not critical.

Thus position $(l'-(m_0-1)k'+1, m_0k')$ is also non-critical by property (Tc). This contradicts the hypothesis about m_0 and *proves the claim*.

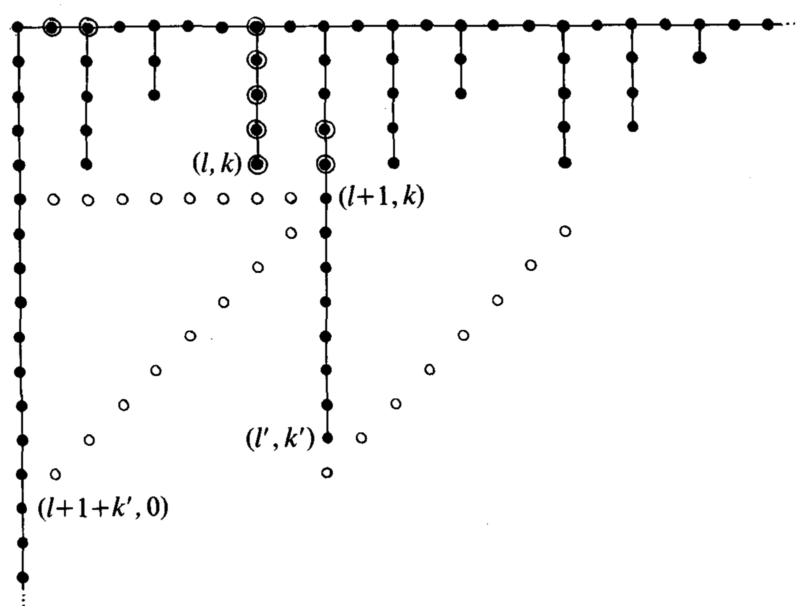


Fig. 4.8

Now we see that the first critical position on or to the right of the position $(l+1, k'+l'-l)$, which exists since the critical end is convergent, is an originating position. Q.E.D. for Lemma 4.6

Proof of Theorem 4.3. (a) If c is periodic, it is clearly recurrent, and has only finitely many originators. So it is convergent by Lemma 4.4.

If c is recurrent and non-periodic, define the function $t: \mathbf{N}^* \to \mathbf{N}$ by the rule that $(t(n), n-t(n))$ is critical and $(m, n-m)$ is not critical for $t(n)<m<n$, so that $\text{mod}(C_{t(n)})=2\ \text{mod}(C_n)$. Lemma 4.6 says exactly that for any level n, there are at least two distinct levels n_1 and n_2 such that $t(n_1)=t(n_2)=n$, so that

$$\text{mod}(C_n) \leqslant \sum_{t(j)=n} \text{mod}(C_j).$$

The function t is strictly decreasing, so we can define the generation of n to be that k such that $t^k(n)=0$. It follows that

$$\sum_{\text{gen}(j)=k} \text{mod}(C_j) \leqslant \sum_{\text{gen}(j)=k+1} \text{mod}(C_j),$$

and Theorem 4.3 (a) follows, since there are infinitely many generations.

(b) Suppose the end $x \in E(R)$ is convergent. For any $l>0$, let $(l, k(l))$ be the first critical position in row l of the tableau $T(x)$ with $k(l) \geqslant 0$. By Lemma 4.4 this position exists. The critical positions $(l, k(l))$ for $l>0$ form the *critical staircase* of the tableau of x. Then

$$\operatorname{mod}(A_{l+k(l),0}(x)) = \operatorname{mod}(C_l).$$

Moreover, the integers $l+k(l)$ are all distinct. Thus

$$\operatorname{mod} N(c) \leqslant \operatorname{mod} N(x) < \infty,$$

which is a contradiction.

(c) If $x \in E(R)$ is precritical it is clearly convergent.

Suppose $x \in E(R)$ is convergent and consider the critical tableau. By (a) it repeats with some period k. There are no critical positions in the tableau beneath row l for some l, except in the columns with column-number divisible by k. Set

$$S = \sum_{i=l+1}^{k+l} \operatorname{mod}(C_i).$$

The critical staircase of the tableau of x is the set of critical positions in $T(x)$ which have no critical positions to the left in the same row.

If $\pi_R^{\circ p}(x) \neq c$ for all p and $\operatorname{mod} N(x) < \infty$, then the staircase of the tableau of x has infinitely many steps. We will show that every step whose tip has vertical coordinate larger than $l+k$ contributes at least S to the $\operatorname{mod} N(x)$; there clearly must be infinitely many of these.

The argument is similar to the one in the proof of Lemma 4.6. If (m, n) is the tip of a step with $m>l+k$, then the diagonal $i+j=m+n+1$ can contain no critical position for $j>l$. Consider the first critical positions to the right of each of

$$(n+m-l, l+1), \ldots, (n+m-l-k+1, l+k).$$

The sum of the moduli of these positions is S. The diagonal drawn from each of these positions to the 0-column ends on an annulus with the same modulus as the critical annulus from which the diagonal left. Since all these endpoints are distinct, it follows that x is divergent and we are done. Q.E.D. for Theorem 4.3

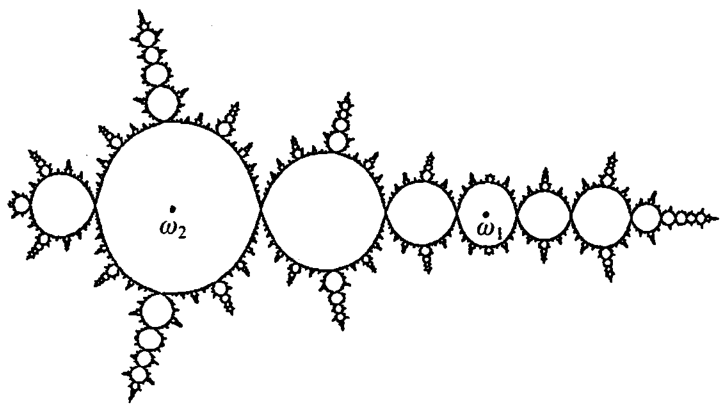


Fig. 5.1

Chapter 5. Julia sets for cubic polynomials

In this chapter we will completely settle the question of when the Julia set of a cubic polynomial is a Cantor set. P. Fatou and G. Julia proved the following result.

THEOREM (Fatou [F], Julia [J]). (a) *The Julia set for a polynomial P is connected if and only if none of the critical points escape to infinity under iteration.*

(b) *The Julia set for a polynomial P is a Cantor set if all the critical points do escape to infinity under iteration.*

Figures 5.1 and 5.2 show a Julia set for a cubic polynomial P satisfying (a) respectively (b) of the theorem.

Fatou conjectured [F, p. 84] that condition (b) above is necessary for the Julia set to be a Cantor set, but this was disproved by H. Brolin [Br], who showed that if P is a real cubic polynomial with one critical point ω_1 escaping to ∞ and the other critical point ω_2 being mapped to a fixed point not in the component of K_P which contains ω_2, then K_P is a Cantor set. Figure 5.3 shows the graph of such a cubic polynomial. The Julia set is a Cantor set on the real axis.

Using tableaux, it is easy to reprove Brolin's result, and more generally to show that when one critical point ω_1 escapes, the other ω_2 does not and the component of K_P containing ω_2 is strictly preperiodic, then the Julia set is always a Cantor set. As we will see this follows from Theorem 4.3 (b) and Lemma 4.4 since in that case the critical end is not recurrent.

Tableaux and originators were invented in order to prove Theorem 5.2 below in the

Fig. 5.2

cases where the critical component is recurrent. As we will see in Chapter 7, this result has consequences in the parameter space also.

In order to show that some Julia sets are not Cantor sets, we will require *polynomial-like mappings* [DH2]. Recall that a polynomial-like mapping of degree d is a triple (U, U', f) where U and U' are plane domains homeomorphic to discs, with U' relatively compact in U, and $f: U' \to U$ is analytic of degree d. The *filled-in Julia set* K_f of the polynomial-like mapping f is defined as

$$K_f = \{z \in U' \mid f^{\circ n}(z) \in U' \text{ for all } n\}.$$

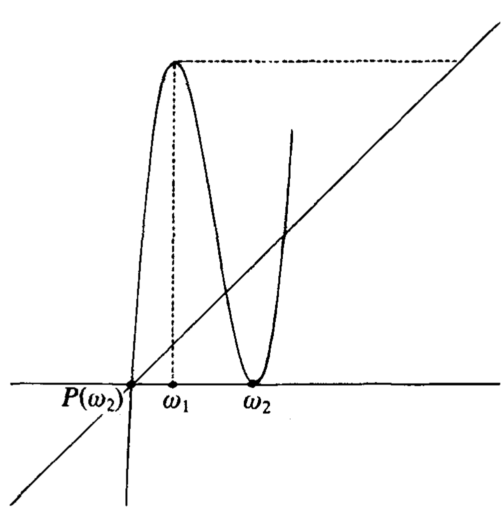


Fig. 5.3

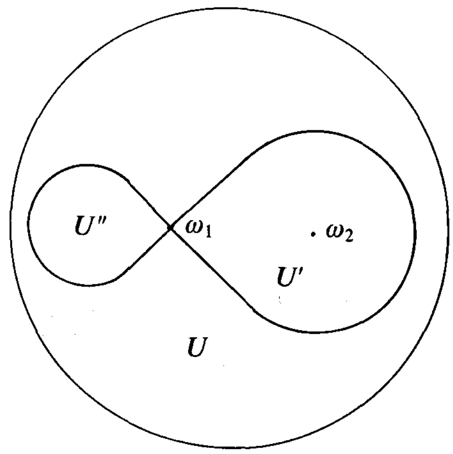


Fig. 5.4

We say that a point z in U' does *not escape* under iteration by f exactly when z is in K_f.

As we will see, there are many iterates of cubic polynomials which are polynomial-like of degree 2; one example stands out particularly. If P is a cubic polynomial with critical points ω_1, ω_2 satisfying $h_P(\omega_2)<h_P(\omega_1)$, then the locus $V=\{z|\,h_P(z)<h_P(\omega_1)\}$ has two connected components, one of which contains ω_2. Call this one U' and the other U''. Let $U=P(U')=\{z|\,h_P(z)<3h_P(\omega_1)\}$. Then $f=P|_{U'}\colon U'\to U$ is polynomial-like of degree 2. See Figure 5.4.

Remark 5.1. It is important to consider the domain of definition U' of a polynomial-like mapping f as part of the definition. For instance, the statement that the critical point of the polynomial-like mapping does not escape is much more restrictive than saying that this critical point has a bounded orbit under P; for example $P(\omega_2)$ might be the fixed point of P which lies in U''; this is precisely what happens in Brolin's example.

We shall need the definition of hybrid equivalence and the straightening theorem from [DH2].

Two polynomial-like mappings (U, U', f) and (V, V', g) of degree d are said to be *hybrid equivalent* if there exists a quasiconformal homeomorphism φ from a neighborhood of K_f onto a neighborhood of K_g, conjugating f and g and so that $\bar{\partial}\varphi=0$ on K_f.

THE STRAIGHTENING THEOREM ([DH2]). (a) *Every polynomial-like mapping (U, U', f) of degree d is hybrid equivalent to a polynomial of degree d.*

(b) *If K_f is connected, then the polynomial is uniquely determined up to conjugation by an affine map.*

5.1. The main statements

Let P be a monic cubic polynomial with one critical point ω_1 escaping to ∞ and the other critical point ω_2 not and with $\lim_{z\to\omega_1'} \varphi_P(z)=\zeta$. Let R_∞ be the associated ζ-pattern and $\bar{\varphi}_P\colon \mathbf{C}-K_P\to R_\infty$ be the analytic isomorphism extending φ_P, as constructed in Corollary 3.5.

The components of the filled-in Julia set K_P for the polynomial P are in 1–1 correspondence with the ends of R_∞. In fact another way of understanding the set $E(\bar{\mathbf{C}}-K_P)$ of ends is as follows: consider the equivalence relation on K_P:

$$x\sim y \quad \text{if and only if} \quad x \text{ and } y \text{ are in the same component of } K_P.$$

The quotient space $K_P/\!\sim$ with the quotient topology is homeomorphic to $E(\bar{\mathbf{C}}-K_P)$; hence it is a Cantor set.

As we shall see, if the critical component of K_P is periodic then the set K_P has countably many components homeomorphic to K_a for some $a\in M$ and uncountably many point components. (Recall that $M=\{a|$ the Julia set of $z\to z^2+a$ is connected$\}$.)

We shall denote the component of K_P corresponding to $x\in E(R_\infty)$ by $K_P(x)$. For any $x\in E(R_\infty)$ we have $P(K_P(x))=K_P(\pi_{R_\infty}(x))$, i.e. the restriction of P to such a component maps onto the image component. This is a consequence of P being proper, with everywhere positive local degree.

We call the component $K_P(x)$ *periodic, preperiodic* or *strictly preperiodic* if the end x has the corresponding property. If x is periodic of period k, the annulus $A_n(x)\in N(x)$ at level n is mapped to $A_{n-k}(x)$. Thus each periodic end, in addition to its period k, has a *level* $n(x)$ which is the smallest integer n such that $A_{n-j}(x)\neq P^{\circ j}(A_n(x))$ for $j=1,2,\dots,k-1$.

The component $K_P(c)$ corresponding to the critical end c is called the *critical component*; of course $\omega_2\in K_P(c)$. Let W_n be the smallest simply-connected domain in $\mathbf{C}$ containing the critical annulus $C_n(P)$. Then each W_n is homeomorphic to a disc and W_n is relatively compact in W_m for each $m<n$.

Theorem 5.2 describes the Julia set if the critical component is not periodic, and Theorem 5.3 describes the Julia set when it is periodic.

Let P be a cubic polynomial with one critical point ω_1 escaping to ∞ and the other ω_2 not.

THEOREM 5.2. *The Julia set J_P is a Cantor set if and only if the critical component is not periodic.*

THEOREM 5.3. *Suppose the critical component is periodic of period k and level n_0.*

(a) *There exists a unique quadratic polynomial of the form $Q(z)=z^2+a$ with $a\in M$, such that the polynomial-like mapping*

$$f=P^{\circ k}|_{W_{n_0}}: W_{n_0}\to W_{n_0-k}$$

is hybrid equivalent to Q.

(b) *A component of K_P is quasi-conformally homeomorphic to K_Q if it is a preimage of the critical component of K_P and is a point otherwise.*

5.2. Analytic preliminaries

In Theorem 5.2 and Theorem 5.3 it is stated precisely which components of K_P are points. In order to prove that we shall need the following two classical results [A]; the proofs in the litterature depend on extremal length, and we will give a different one.

PROPOSITION 5.4 (Grötzsch). *Let (A_j) be a (finite or infinite) sequence of open annuli. Let $\varphi_j: A_j\to A$ be conformal mappings which are homotopy equivalences, with disjoint images. Then*

$$\sum \operatorname{mod}(A_j)\le \operatorname{mod} A.$$

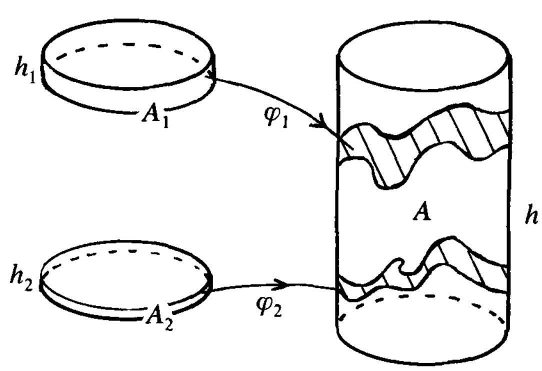


Fig. 5.5

Proof. Let $B(h)=\{z|0<\mathrm{Im}(z)<h\}$, and write

$$A=B(h)/\mathbf{Z}, \quad A_j=B(h_j)/\mathbf{Z}$$

so that h and h_j are the moduli of the corresponding annuli. Parametrize the annuli by $z=x+iy$, $z_j=x_j+iy_j$. Then

$$h=\int_A dx\,dy \geqslant \sum_j \int_{\varphi_j(A_j)} dx\,dy=\sum_j \int_{A_j} |\varphi_j'(z_j)|^2\,dx_j\,dy_j=\sum_j \int_0^{h_j}\left(\int_0^1 |\varphi_j'(z_j)|^2\,dx_j\right)dy_j$$
$$\geqslant \sum_j \int_0^{h_j}\left(\int_0^1 |\varphi_j'(z_j)|\,dx_j\right)^2 dy_j \geqslant \sum_j \int_0^{h_j} dy_j \geqslant \sum h_j.$$

The first inequality follows from the fact that the A_j are disjoint, the second is Schwarz's inequality, and the third is that the image of all "equators" of annuli A_j have length greater than 1, since the inclusions φ_j are homotopy equivalences.

Q.E.D. for Proposition 5.4

PROPOSITION 5.5. *Let $A\subset D$ be an open annulus of infinite modulus. Then the bounded component of $\mathbf{C}-A$ is a point.*

Proof. If the component contained two points x, y, then the geodesic in $D-\{x,y\}$ (for its hyperbolic metric) in the homotopy class of A would be shorter than any curve in A in that homotopy class. However, there are arbitrarily short curves in A in that homotopy class. Q.E.D. for Proposition 5.5

5.3. Proofs of the theorems

Proof of Theorem 5.2. Using Theorem 4.3 (a) we have that the following three statements are equivalent: $P^{\circ n}(\omega_2)$ is not in $K_P(c)$ for any $n>0$, the critical end $c\in E(R)$ is non-periodic, the critical end is divergent. If this is satisfied it follows from Theorem 4.3 (b) and Propositions 5.4 and 5.5 that all components of K_P are just points. This proves that non-periodicity of the critical end implies that K_P is a Cantor set. Hence the Julia set $J_P=K_P$ is a Cantor set. The converse is contained in Theorem 5.3, proved below. Q.E.D. for Theorem 5.2, $\Leftarrow$

Remark 5.6. If the critical component is strictly preperiodic, then it is non-recurrent. Thus we do not need the full power of Theorem 4.3 (a) for the proof: the critical nest is divergent by Lemma 4.4, and all others also by Theorem 4.3 (b). Hence in that case the Julia set is a Cantor set, reproving Brolin's result in greater generality. Notice that the proofs of Theorem 4.3(b) and Lemma 4.4 do not require the third condition on tableaux.

In the case where the critical component is periodic we know that the critical end is convergent. However, Grötzsch' inequality in Proposition 5.4 goes in the wrong direction, and would still allow $K_P(c)$ to be a point. That this is in fact not the case follows from the theory of polynomial-like mappings.

PROPOSITION 5.7. (a) *Suppose there exist n and k with* $k \leqslant n$ *such that P satisfies* $\operatorname{depth}(P) \geqslant n$, *that* $P^{\circ k}(C_n(P)) = C_{n-k}(P)$ *and that* $P^{\circ j}(C_n(P)) \neq C_{n-j}(P)$ *for* $0<j<k$. *Then the restriction*

$$f_n = P^{\circ k}|_{W_n} \colon W_n \to W_{n-k}$$

is polynomial-like of degree 2.

(b) *If the critical end is periodic of period k and level* n_0, *then the hypotheses above hold for all* $n \geqslant n_0$ *and the critical point* ω_2 *does not escape for the polynomial-like mapping* f_{n_0}.

Proof. Clearly f_n maps the boundary ∂W_n to the boundary ∂W_{n-k}. The mapping $P\colon W_n \to P(W_n)$ is of degree 2, and all the other restrictions of P which appear in f_n are of degree 1. This verifies (a).

For (b), the same argument as above shows that f_n is polynomial-like for all $n \geqslant n_0$; this implies that the image $f_{n_0}(\omega_2)$ lies in $\bigcap_n U_n = K(f_{n_0})$. Q.E.D. for Proposition 5.7

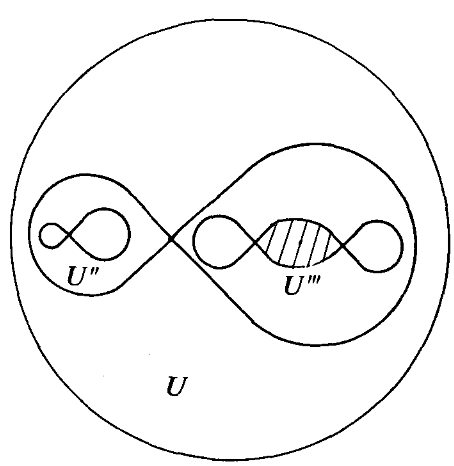


Fig. 5.6

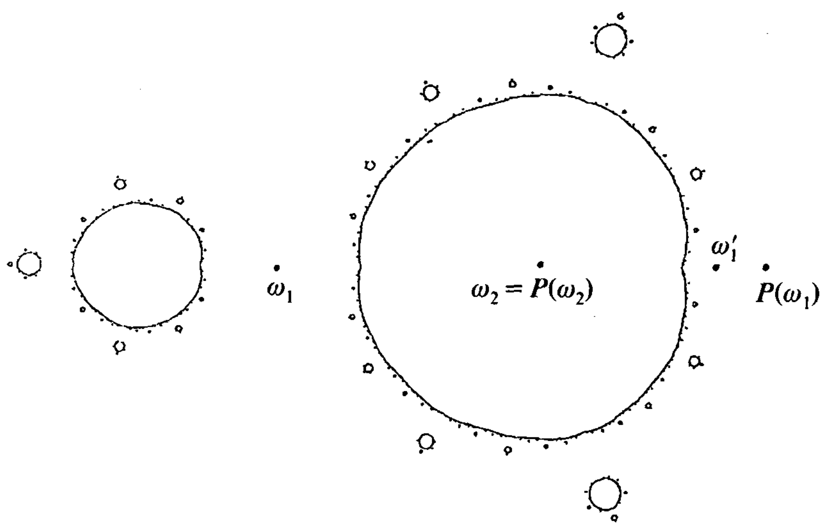


Fig. 5.7

Example 5.8. Suppose a polynomial P has depth >1, and that the critical value lies in the small lobe U'' (cf. Figure 5.4). Set U''' to be the component of $P^{-1}(U'')$ contained in U'. Then $P^{\circ 2}\colon U''' \to U$ is polynomial-like of degree 2. (See Figure 5.6.)

Proof of Theorem 5.3. Part (a) follows immediately from Proposition 5.7 and the straightening theorem.

Now consider a convergent end $x \in E(R)$. Let n be the smallest integer such that $\pi_R^{\circ n}(x) = c$; the existence of such an n is guaranteed by Theorem 4.3(b). Then $P^{\circ n}$ maps $K_P(x)$ onto $K_P(c)$ homeomorphically. A component $K_P(x)$ for which the end $x \in E(R)$ is divergent is of course just a point (Propositions 5.4 and 5.5). Q.E.D. for Theorem 5.3

Q.E.D. for Theorem 5.2

Examples of Julia sets J_P for some polynomials P with periodic critical component are shown in Figures 5.7–5.12. In Figures 5.7, 5.8, 5.9 the polynomial P satisfies $P(\omega_2) = \omega_2$ and $\arg(\bar{\varphi}_P(\omega_1'))$ equals 0, 1/3, 2/3 respectively. The unique quadratic polynomial associated to P due to Theorem 5.5(a) is $Q(z) = z^2$.

In Figures 5.10, 5.11 the polynomial P satisfies $P^{\circ 2}(\omega_2) = \omega_2$, $P(\omega_2) \neq \omega_2$, $\arg(\bar{\varphi}_P(\omega_1')) = 0$ and $P(\omega_2)$ belongs to U', U'' respectively (with the notation used in Figures 5.3 and 5.4). The critical component of the polynomial P in Figure 5.10 has period 1 and the unique quadratic polynomial associated to P is $Q(z) = z^2 - 1$. The critical

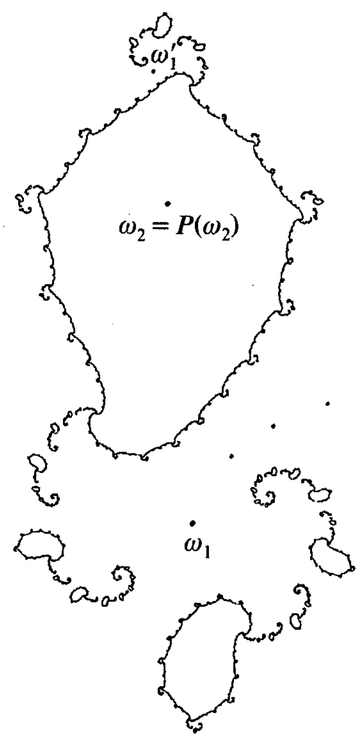


Fig. 5.8

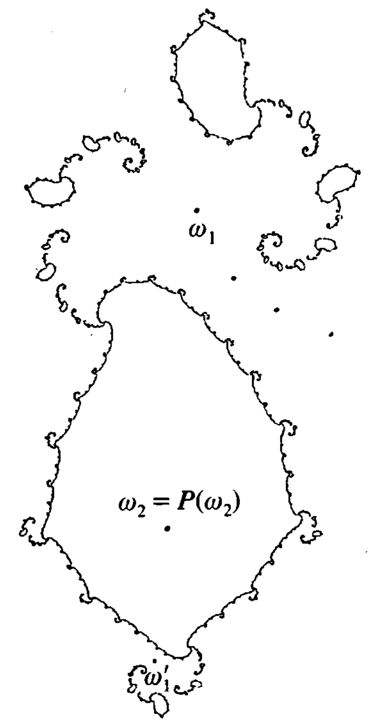


Fig. 5.9

component of the polynomial P in Figure 5.11 has period 2 and the unique quadratic polynomial associated to P is $Q(z)=z^2$.

In Figure 5.12 the polynomial P satisfies $P^{\circ 2}(\omega_2)$ is fixed, $P(\omega_2) \neq P^{\circ 2}(\omega_2)$, $\arg(\bar{\varphi}_P(\omega_2'))=0$ and the period of the critical component is 1. The unique quadratic

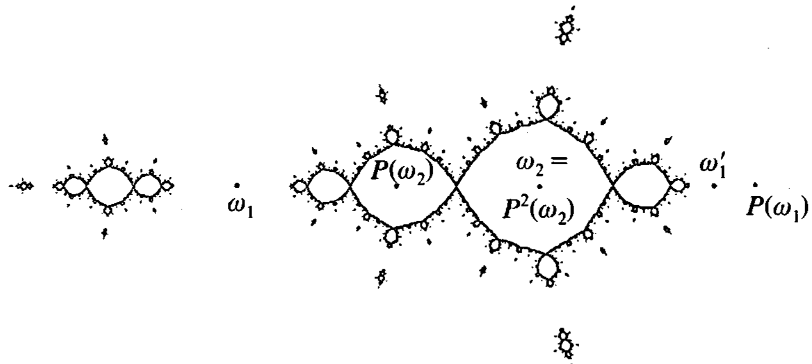


Fig. 5.10

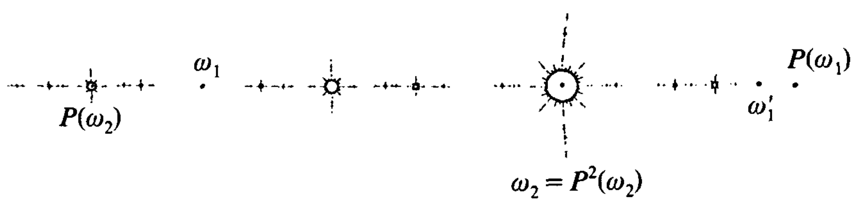


Fig. 5.11

polynomial associated to P is $Q(z)=z^2-2$. The polynomials used in Figures 5.7, 5.10, 5.11 and 5.12 are all real.

5.4. The measure of Julia sets

In this section, we will show that those Julia sets which are Cantor sets by Theorem 5.2 are of measure zero. The proof is due to Curt McMullen, who kindly agreed to let us include it in our paper.

THEOREM 5.9. *If P is a polynomial with non-periodic critical component then the Julia set is a Cantor set of measure* 0.

First, a preliminary lemma, very similar to Proposition 5.4.

PROPOSITION 5.10. *Let $D\subset\mathbf{C}$ be a simply connected open set, $K\subset D$ a connected compact subset and A the annulus $D-K$. Then*

$$4\pi \operatorname{mod}(A) \leqslant \frac{\operatorname{area}(D)}{\operatorname{area}(K)} - 1.$$

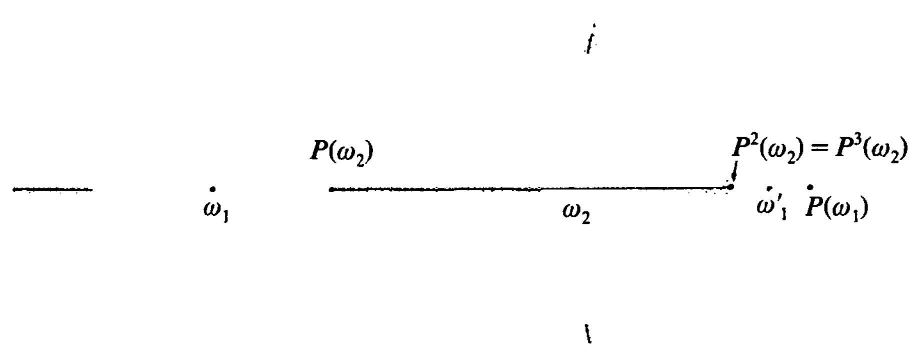


Fig. 5.12

Proof. If K has area zero, the proposition is trivially true. Suppose area$(K)\neq 0$ and scale so that area$(K)=1/4\pi$, which is the area of the disc bounded by the circle of circumference 1.

Let $B(h)=\{z|\,0<\mathrm{Im}(z)<h\}$, and $B=B(h)/\mathbf{Z}$ so that if $h=\mathrm{mod}(A)$, there exists a conformal isomorphism $\varphi: B\to A$. Let $z=x+iy$ be the natural coordinate in D and $w=u+iv$ the natural coordinate in B, we have

$$\begin{aligned}\mathrm{area}(A) &= \int_A dx\,dy = \int_B |\varphi'(w)|^2\,du\,dv = \int_0^h \left(\int_0^1 |\varphi'(w)|^2\,du\right) dv\\ &\geqslant \int_0^h \left(\int_0^1 |\varphi'(w)|\,du\right)^2 dv \geqslant h,\end{aligned}$$

where the first inequality is Schwarz's inequality, and the second comes from the isoperimetric inequality: any closed curve which surrounds K has length $\geqslant 1$.

This can be rewritten area$(D)-$area$(K)\geqslant$mod(A) when area$(K)=1/4\pi$, or

$$\frac{\mathrm{area}(D)-\mathrm{area}(K)}{\mathrm{area}(K)}\geqslant 4\pi\,\mathrm{mod}(A).$$

In this form, both sides are scale-independent, proving the proposition.

Q.E.D. for Proposition 5.10

Remark 5.11. (a) The inequality in Proposition 5.10

$$\mathrm{area}(D)\geqslant \mathrm{area}(K)\,(1+4\pi\,\mathrm{mod}(A))$$

can be improved to give

$$\mathrm{area}(D)\geqslant \mathrm{area}(K)\,e^{4\pi\,\mathrm{mod}(A)}.$$

For each n divide the annulus A into a nested sequence of n annuli, all of modulus mod$(A)/n$. For each such annulus A_i let D_i be the disc bounded by the outer boundary of A_i. We obtain a sequence of compact sets $K=K_0\subset K_1=\bar{D}_1\subset\ldots\subset K_{n-1}=\bar{D}_{n-1}$ in $D_n=D$ with

$$\mathrm{area}(D_{i+1})\geqslant \mathrm{area}(K_i)\,(1+4\pi\,\mathrm{mod}(A)/n)\quad\text{for}\quad i=0,1,\ldots,n-1.$$

Therefore area$(D)\geqslant$area$(K)\,(1+4\pi\,\mathrm{mod}(A)/n)^n$ for all n and the improved inequality follows.

(b) One can prove that area$(D)=$area$(K)\,e^{4\pi\,\mathrm{mod}(A)}$ if and only if K and D are exact concentric discs. (It does not follow from the proof given above.)

Now consider a cubic polynomial with one critical point escaping and the other not, and with the nests of all ends of $\mathbf{C}-K_P$ divergent. Let $\mathscr{A}_N$ be the set of annuli at level N, and for each such annulus A, let D_A be the disc bounded by its outer boundary. For each $A \in \mathscr{A}_N$, let $u(A) \in \mathscr{A}_{N-1}$ be the annulus at level $N-1$ such that $A \subset D_{u(A)}$ and set

$$\mu(A) = \frac{\operatorname{area}(D_A)}{\sum_{A' \in u^{-1}(A)} \operatorname{area}(D_{A'})}.$$

Define $\nu(A)$ inductively by

$$\nu(A) = \nu(u(A))\,\mu(A),$$

starting the induction by $\nu(A_0)=\mu(A_0)$ where A_0 is the unique element of $\mathscr{A}_0$.

Let $c_N = \inf_{A \in \mathscr{A}_n} \nu(A)$; for any $A \in \mathscr{A}_N$ we have $\nu(u(A))\,\mu(A) \geqslant c_N$.

LEMMA 5.12. *The sequence of numbers* $c_0, c_1, c_2, \ldots$ *tends to* ∞.

Proof. The opposite statement is that there exists a constant c such that for all N there is a product as above which is smaller than c. Since the sets $\mathscr{A}_N$ are finite, we can choose by a diagonal argument a sequence of annuli A_i all belonging to the same nest, and such that the infinite product $\Pi\,\mu(A_i) \leqslant c$. In that case there exists a constant C such that

$$\log \mu(A_i) \geqslant C(\mu(A_i)-1) \quad \text{for all } i,$$

and then

$$\log c \geqslant \sum \log \mu(A_i) \geqslant C \sum (\mu(A_i)-1) \geqslant 4\pi C \sum \operatorname{mod}(A_i)$$

which contradicts the hypothesis that all nests are divergent. Q.E.D. for Lemma 5.12.

Proof of Theorem 5.9. First observe

$$\begin{aligned}\sum_{A' \in \mathscr{A}_N} \nu(u(A'))\operatorname{area}(D_{A'}) &= \sum_{A \in \mathscr{A}_{N-1}} \nu(A) \sum_{A' \in u^{-1}(A)} \operatorname{area}(D_{A'}) \\ &= \sum_{A \in \mathscr{A}_{N-1}} \nu(A)\operatorname{area}(D_A)/\mu(A) \\ &= \sum_{A \in \mathscr{A}_{N-1}} \nu(u(A))\operatorname{area}(D_A).\end{aligned}$$

Hence $\Sigma\, \nu(u(A'))\,\mathrm{area}(D_{A'})$ is a constant C independent of N. Moreover

$$\begin{aligned}\sum_{A'\in\mathcal{A}_N} \mathrm{area}(D_{A'}) &= \sum_{A\in\mathcal{A}_{N-1}} \mathrm{area}\,(D_A)/\mu(A)\\ &\leqslant c_{N-1}^{-1} \sum_{A\in\mathcal{A}_{N-1}} \nu(u(A))\,\mathrm{area}(D_A)\\ &= Cc_{N-1}^{-1}.\end{aligned}$$

Therefore $\Sigma\,\mathrm{area}(D_A)$ tends to 0 as N tends to ∞ by Lemma 5.12.

Q.E.D. for Theorem 5.9

Remark 5.13. This proof can be adapted to show that the Julia set of many other polynomials P have measure 0 when they are Cantor sets. The only information we need is that the critical level curves of h_p define divergent nests. This occurs for instance in any degree when all critical points escape. The result is known in that case (see for instance [DH1] p. 40), but the previous proof is quite delicate. This proof appears to be simpler, since in this case the moduli of the annuli of nests are bounded below.

We will see in Chapter 12 some other examples of Julia sets of measure 0 in higher degrees; they are cases where the tableau argument applies.

Chapter 6. Monodromy of patterns

6.1. The local trivialization of the pattern bundle

The bundle $p_N\colon U_N\to\mathbf{C}-\bar{D}$ defined in Section 2.8 is of course not analytically locally trivial, since we have seen in Proposition 2.9 that only two different patterns are isomorphic.

On the other hand, $p_N\colon U_N\to\mathbf{C}-\bar{D}$ is *topologically* locally trivial, and we will now proceed to give an explicit local trivialization. The construction is essentially an extension of the "wring construction" of [BH].

In Section 8 of that paper we defined a family of complex structures σ_u on $\mathbf{C}-\bar{D}$, with corresponding Beltrami forms $\mu_u(z)$, parametrized by $u=s+it$ in the right half-plane $s>0$ and given by the formula

$$\mu_u(z)=\frac{u-1}{u+1}\frac{\bar{z}}{z}\frac{\overline{dz}}{dz}.$$

The map $f_u\colon \mathbf{C}-\bar{D}\to\mathbf{C}-\bar{D}$ giving by $z\mapsto z|z|^{u-1}$ is a diffeomorphism commuting with $z\mapsto z^n$ for any $n>0$, and satisfies

$$\frac{\bar\partial f_u}{\partial f_u}=\mu_u,$$

i.e. it integrates the complex structure σ_u.

Take a Riemann surface $R\in\mathcal{P}_N(\zeta)$, and consider the complex structure on R_N obtained as follows. Let

$$R_n=\left\{z\in R \,\middle|\, h_R(z)>\frac{\log|\zeta|}{3^n}\right\};$$

on $R_0=\mathbf{C}-\bar{D}_{|\zeta|}$ put the structure σ_u, then on R_1 put $\pi_1{}^*(\sigma_u)$, etc. Call the resulting Riemann surfaces (R_n,σ_u), so $(R,\sigma_u)=(R_N,\sigma_u)$. Since (R,σ_u) is obtained from (R_1,σ_u) by a succession of standard triple covers, exactly as any pattern, it must be uniquely isomorphic to a unique element $R(u)\in\mathcal{P}_N(\zeta')$ for $\zeta'=f_u(\zeta)=\zeta|\zeta|^{u-1}$. Call the isomorphism above $f_{R,u}\colon(R,\sigma_u)\to R(u)$; clearly it extends f_u on R_0.

Remark 6.1. Let $R\in\mathcal{P}_N(\zeta)$. For each $0\le n\le N$ the isomorphism $f_{R,u}$ maps the critical annulus $C_n(R)$ onto the critical annulus $C_n(R(u))$ and for each $0\le n<N$ the critical value annulus $B_{n-1}(R)$ onto the critical value annulus $B_{n-1}(R(u))$. Moreover for any annulus A of R nested inside an annulus A' the annulus $f_{R,u}(A)$ of $R(u)$ is nested inside the annulus $f_{R,u}(A')$.

Choose ζ, and choose U a neighborhood of 1 in the right half-plane so that the mapping $U\to\mathbf{C}-\bar{D}$ given by $u\mapsto\zeta|\zeta|^{u-1}$ is injective; let V be its image. Set $p_N^{-1}(\{\zeta\})=U_N(\zeta)$.

PROPOSITION 6.2. (a) *The mapping $U_N(\zeta)\times U\to p_N^{-1}(V)$ defined by $(z,u)\mapsto f_{R,u}(z)$ for $z\in R\in\mathcal{P}_N(\zeta)$ and $u\in U$ is a homeomorphism, and the diagram*

$$\begin{array}{ccc} U_N(\zeta)\times U & \xrightarrow{f_{R,u}} & p_N^{-1}(V) \\ \big\downarrow & & \big\downarrow \\ U & \xrightarrow{u\mapsto\zeta|\zeta|^{u-1}} & V \end{array}$$

commutes. In particular, $p_N\colon U_N\to\mathbf{C}-\bar{D}$ is topologically a locally trivial fibration.

(b) *If V_1 and V_2 are two simply connected open subsets of* $\mathbf{C}-\bar{D}$ *with* $V_1 \cap V_2$ *connected and* $\zeta \in V_1 \cap V_2$, *then the local trivializations given in* (a) *coincide above* $V_1 \cap V_2$.

6.2. The monodromy of the pattern bundle

A locally trivial bundle over a circle, and hence also over $\mathbf{C}-\bar{D}$, is topologically determined by the isotopy class of its monodromy. In particular, the locally trivial bundle $p_N: U_N \to \mathbf{C}-\bar{D}$ is determined by its monodromy, which is a homeomorphism of $U_N(\zeta)$, only defined up to isotopy.

Proposition 6.2(b) says that the local trivializations induced by wringing the complex structure fit together to give a trivialization of the pull back of $p_N: U_N \to \mathbf{C}-\bar{D}$ to the universal cover of $\mathbf{C}-\bar{D}$, hence computes this monodromy: it is the map which sends $z \in R \in \mathcal{P}_N(\zeta)$ to $f_{R,u}(z) \in R(u)$, when $\zeta|\zeta|^{u-1}=\zeta$, i.e. when $u(\zeta)=1+2\pi i/\log|\zeta|$. More precisely, choose some $r>1$ as base point in $\mathbf{C}-\bar{D}$, to get:

THEOREM 6.3. (a) *For all $n \geq 0$ there exists unique homeomorphisms $m_n: U_n(r) \to U_n(r)$ such that the diagrams*

$$\begin{tikzcd} U_n(r) \arrow[r, "m_n"] \arrow[d, "\pi_n"'] & U_n(r) \arrow[d, "\pi_n"] \\ U_{n-1}(r) \arrow[r, "m_{n-1}"] & U_{n-1}(r) \end{tikzcd}$$

commute, and $m_0: U_0(r) \to U_0(r)$ is the map $f_u: \mathbf{C}-\bar{D}_r \to \mathbf{C}-\bar{D}_r$ for

$$u = 1+2\pi i/\log r.$$

(b) *The monodromy $m_N: U_N(r) \to U_N(r)$ of the bundle $p_N: U_N \to \mathbf{C}-\bar{D}$ induced by the canonical local trivializations above is the mapping m_N.*

Proof. Going along the vertical line $u=1+it$ and starting with any $R \in U_N(r)$, the path $t \mapsto p_N(R(u(t)))=r^{u(t)}$ in $\mathbf{C}-\bar{D}$ returns to r for the first time for $u=1+2\pi i/\log r$, and forms a loop which generates $\pi_1(\mathbf{C}-\bar{D})$. So f_u with this value of u is the monodromy of the bundle $p_N: U_N \to \mathbf{C}-\bar{D}$. Q.E.D. for Theorem 6.3

Let $R\in\mathcal{P}_N(r)$ and let A be any annulus of R at level $0\leq n\leq N$. Define the *period* of A *under the monodromy* m_N to be the smallest k such that $m_N^{\circ k}(A)=A$. It follows from Remark 6.1 and Theorem 6.3 that the period under m_N of the critical annuli $C_n(R)$ and the critical value annuli $B_{n-1}(R)$ is the same number. Moreover any other annulus A of R has a period under m_N which is a multiple of that.

Proposition 6.4. (a) *Let $R\in\mathcal{P}_N(r)$. The periods under the monodromy m_N of a nested sequence of annuli $A_0, A_1, \ldots, A_N$ of R form an increasing sequence of powers of* 2.

(b) *Let $R\in\mathcal{P}_\infty(r)$. Suppose the critical end is periodic of period k and level n_0; i.e. $\pi_R^{\circ k}(C_N(R))=C_{N-k}(R)$ for all $N\geq n_0$ and $\pi_R^{\circ j}(C_N(R))\neq C_{N-j}(R)$ for $0<j<k$. Then for all $N\geq n_0$ the period of $C_N(R)$ and $B_{N-1}(R)$ under the monodromy m_N is equal to the period of $C_{n_0}(R)$ under the monodromy m_{n_0}.*

Proof. (a) If an annulus A is nested inside A' then the period of A under m_N is of course a multiple of the period of A', so we only need to prove that the period of any annulus A of R at level $0\leq n\leq N$ is a power of 2. The proof is by induction on N.

For $N=1$ there is a unique pattern $R_1\in\mathcal{P}_1(r)$. The period under m_1 of the annulus at level 0 and the two annuli at level 1 equals 1.

Suppose the statement is true for any pattern in $\mathcal{P}_{N-1}(r)$. We shall prove it true for any $R\in\mathcal{P}_N(r)$.

From $\pi_N(C_N(R))=i_R^{-1}(B_{N-1}(R))$ in $s_N(R)$ and $\pi_N\circ m_N=m_{N-1}\circ\pi_N$ we conclude that the period of $C_N(R)$ under m_N equals the period of $i_R^{-1}(B_{N-1}(R))$ under m_{N-1}. From the induction hypothesis we know that the period of $i_R^{-1}(B_{N-1}(R))$ under m_{N-1} is a power of 2. Denote this period by $k(C_N(R))$.

Let A be any annulus of R different from $C_N(R)$ at level $0<n\leq N$, and let A' denote the annulus at level $n-1$ in which A is nested. The annulus $\pi_N(A)=\bar{A}$ in $s_N(R)$ is nested inside $\pi_N(A')=\bar{A}'$. Let $k(\bar{A})$ and $k(i_R^{-1}(A'))$ denote the period of $\bar{A}$ and $i_R^{-1}(A')$ under m_{N-1} respectively. The number

$$k=\sup\big(k(C_N(R)), k(\bar{A}), k(i_R^{-1}(A'))\big)$$

is a power of 2. Using $\pi_N\circ m_N=m_{N-1}\circ\pi_N$ we see that k is the smallest number such that the annulus $m_N^{\circ k}(A)$ is nested inside A' and mapped to $\bar{A}$ under π_N. If $A'\neq C_{n-1}(R)$ then the annulus A is the only one with this property. If $A'=C_{n-1}(R)$ then there are two annuli with this property. If $m_N^{\circ k}(A)\neq A$ then the period of A under m_N equals $2k$, in all other cases it equals k.

(b) This follows immediately by inspection from the proof of (a).

Q.E.D. for Proposition 6.4

6.3. The monodromy of the quotient pattern bundle

The monodromy of the quotient bundle $\bar{P}_N\colon \bar{U}_N \to \mathbf{C}-\bar{D}$ described in Section 2.9 is a "functional square root" of the mapping m_N above, and actually rather more natural than m_N.

We chose $r>1$ as base point in $\mathbf{C}-\bar{D}$ for the bundle $p_N\colon U_N \to \mathbf{C}-\bar{D}$, so it is natural to choose $r^2 \in \mathbf{C}-\bar{D}$ as the base point for $p_N\colon U_N \to \mathbf{C}-\bar{D}$ since $U_N(r)$ is naturally isomorphic to $\bar{U}_N(r^2)$.

THEOREM 6.5. (a) *For all $n \geq 0$ there exist unique homeomorphisms $\tilde{m}_n\colon U_N(r) \to U_N(r)$ such that the diagrams*

$$\begin{tikzcd} U_n(r) \arrow[r, "\tilde{m}_n"] \arrow[d, "\pi_n"'] & U_n(r) \arrow[d, "\pi_n"] \\ U_{n-1}(r) \arrow[r, "\tilde{m}_{n-1}"] & U_{n-1}(r) \end{tikzcd}$$

commute, and $\tilde{m}_0\colon U_0(r) \to U_0(r)$ is the map $\mathbf{C}-\bar{D}_r \to \mathbf{C}-\bar{D}_r$ given by $z \mapsto -f_u(z)$ for

$$u = 1 + \pi i/\log r.$$

(b) *The monodromy $\tilde{m}_N\colon \bar{U}_N(r^2) \to \bar{U}_N(r^2)$ of the bundle $\bar{p}_N\colon \bar{U}_N \to \mathbf{C}-\bar{D}$ is the mapping $\tilde{m}_N$.*

(c) *The monodromy m_N of $p_N\colon U_N \to \mathbf{C}-\bar{D}$ is given by $(\tilde{m}_N)^{\circ 2}$.*

Proof. Going along the vertical line $u=1+it$ and starting with any $R \in \bar{U}_N(r^2)$, the path $t \mapsto p_n(R(u(t))) = r^{2u(t)}$ in $\mathbf{C}-\bar{D}$ returns to r^2 for the first time for $u=1+\pi i/\log r$, and forms a loop which generates $\pi_1(\mathbf{C}-\bar{D})$. So $-f_u$ with this value of u is the monodromy of the bundle $\bar{p}_N\colon \bar{U}_N \to \mathbf{C}-\bar{D}$. Q.E.D. for Theorem 6.5

As for the monodromy m_N we can define the period of an annulus A of a pattern R under the monodromy $\tilde{m}_N$. We get a proposition analogous to Proposition 6.4.

PROPOSITION 6.6. (a) *Let $R \in \mathcal{P}_N(r)$. The periods under the monodromy $\tilde{m}_N$ of a nested sequence of annuli $A_0, A_1, \ldots, A_N$ of R form an increasing sequence of powers of 2.*

(b) *Suppose $R \in \mathcal{P}_\infty(r)$ and the critical end is periodic of period k and level n_0. Then for all $N \geq n_0$ the period of $C_N(R)$ and $B_{N-1}(R)$ under the monodromy $\tilde{m}_N$ is equal to the period of $C_{n_0}(R)$ under $\tilde{m}_{n_0}$.*

6.4. Fractional Dehn twists

It is possible to be more explicit about the mapping $\tilde{m}_N$, in terms of fractional Dehn twists.

Suppose we are given annuli A and A' of the same modulus M, both with an "inner" and an "outer" boundary. By the standard conformal representation of annuli onto regions

$$\{z \mid e^{-2\pi M} \leq |z| \leq 1\},$$

we can consider both outer boundaries as metric circles. Suppose we are given an orientation preserving isometry f from the outer boundary of A to the outer boundary of A'. The map f can be extended to the *fractional Dehn twist of fraction* q as follows: identify both A and A' to the standard annulus as above so that f becomes the identity, and consider the map

$$\varrho e^{i\theta} \mapsto \varrho e^{i(\theta + q(\log \varrho)/M)}.$$

See Figure 6.1. The definition makes sense for any real number q, but in the following q will always be of the form 2^{-k} for some $k \geq 0$.

We shall assign to each annulus A of each $R \in \mathcal{P}_N(\zeta)$ the *fraction* of A

$$q(A) = 2\pi \frac{\operatorname{mod}(A)}{\log |\zeta|^2} = \frac{\operatorname{mod}(A)}{\operatorname{mod}(C_0)}.$$

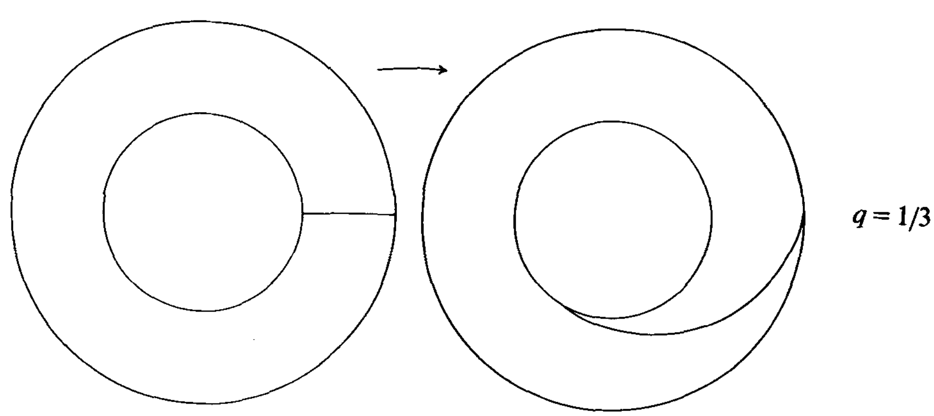
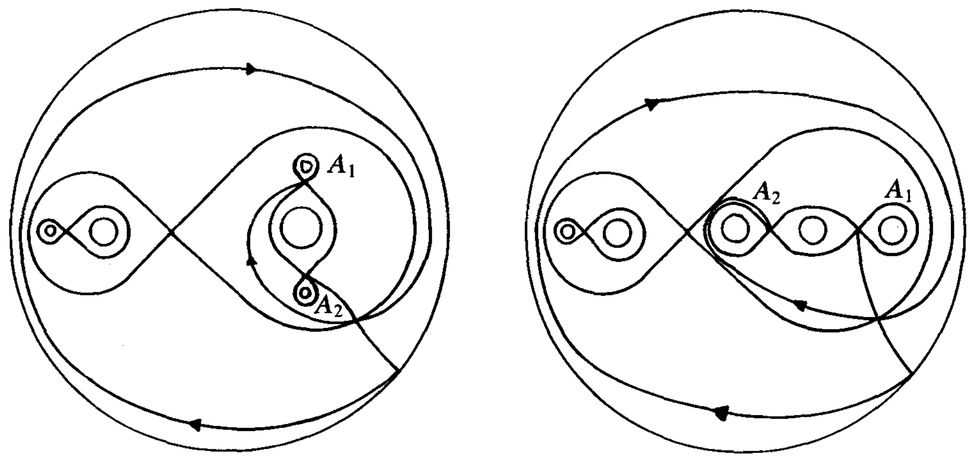


Fig. 6.1



(a) (b)

Fig. 6.2

This can also be done recursively by setting $q(A)=1$ for the unique annulus C_0 of R, and then for any annulus A by setting

$$q(A)=\frac{q(\pi_R(A))}{\deg(\pi_R|_A)}.$$

THEOREM 6.7. *The monodromy $\bar{m}_N$ of the bundle $\bar{p}_N\colon \bar{U}_N \to \mathbf{C}-\bar{D}$ can be defined recursively as follows:*

(a) *On $\bar{U}_0(r^2)$ the map $\bar{m}_0$ is the map $\mathbf{C}-\bar{D}_r \to \mathbf{C}-D_r$ given by $z \mapsto -f_u(z)$ for $u=1+\pi i/\log r$. More explicitely, it maps $z=\varrho e^{i\theta}$ to $-z|z|^{u-1}=-\varrho e^{i(\theta+\pi\log\varrho/\log r)}$.*

(b) *Any pattern $R\in\mathcal{P}_{n+1}(r)$ has a critical value annulus $B_n(R)$; R is mapped to the R' corresponding to the critical value annulus $B_n(R')=\bar{m}_n(B_n(R))$; if A is an annulus of $R-i_R(s_{n+1}(R))$ with outer boundary γ then A is mapped to the annulus A' of R' with outer boundary $\gamma'=\bar{m}_n(\gamma)$ by the extension of $\bar{m}_n|_\gamma$ as a fractional Dehn twist of fraction $q(A)$.*

Proof. Statement (a) is already included in Theorem 6.5. Statement (b) is proved by induction on n. The mapping $\bar{m}_{n+1}$ must make the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\bar{m}_{n+1}} & A' \\ {\scriptstyle \pi_R}\downarrow & & \downarrow{\scriptstyle \pi_{R'}} \\ \pi_R(A) & \xrightarrow{\bar{m}_n} & \pi_{R'}(A') \end{array}$$

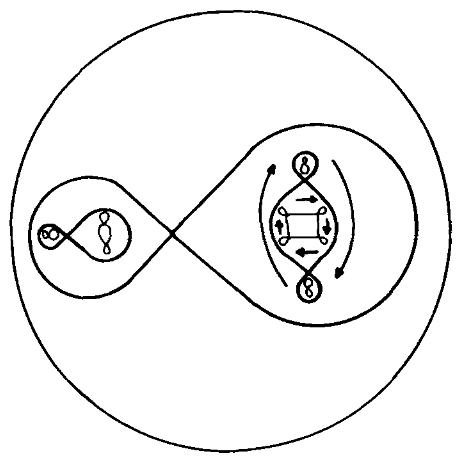


Fig. 6.3

If $\bar{m}_n: \pi_R(A) \to \pi_{R'}(A')$ is a fractional Dehn twist of fraction q, then $\bar{m}_{n+1}: A \to A'$ must be a fractional Dehn twist of fraction $q/\deg(\pi_R)$. The numbers $q(A)$ were defined so as to have this property. Q.E.D. for Theorem 6.7

Figure 6.2 shows the images $\bar{m}_2(A)$ of the annuli at level 2 for both patterns $R \in \mathcal{P}_2(r)$. In both cases the annuli A_1 and A_2 are interchanged. Figure 6.3 shows the images $\bar{m}_3(A)$ of the annuli at level 3 for a particular $R \in \mathcal{P}_3(r)$.

Chapter 7. Parapatterns

Our next object is to describe the parameter space for patterns. As it turns out, practically all the work has already been done.

7.1. Parapatterns for fixed ζ

Each pattern $R \in \mathcal{P}_N(\zeta)$, $N>0$, has a deepest critical annulus $C_N(R)$, and a deepest critical value annulus $B_{N-1}(R)$. The *ζ-parapattern* is defined as the quotient of the disjoint union

$$\Omega(\zeta) = \left[\bigcup_{N \geq 1} \bigcup_{R \in \mathcal{P}_N(\zeta)} \overline{B_{N-1}(R)} \right] \Big/ \sim$$

with the equivalence identifying for each $N>1$ and for each $R \in \mathcal{P}_N(\zeta)$, the point x in the outer boundary of $B_N(s_N(R))$ to $i_R(x)$ in the inner boundary of $B_{N-1}(R)$. It is quite possible to give this space the structure of a Riemann surface directly, but the gluing takes place on a graph and this makes the construction a little fussy. There is an alternative equivalent definition where all the gluing takes place on open sets.

Each R contains the sub-Riemann surface $V(R)$ consisting of the annulus $B_{N-1}(R)$ and all points inside it, and $V'(R)=V(R)-\overline{B_{N-1}(R)}$, which is a disjoint union of annuli as defined in Section 2.8. Recall that $\mathcal{P}_1(\zeta)$ has a unique element R_1. We can also define $\Omega(\zeta)$ to be the union

$$\Omega(\zeta)=\left[\overline{V(R_1)}\cup\bigcup_{N>1}\bigcup_{R\in\mathcal{P}_N(\zeta)}V(R)\right]\Big/\sim$$

with the equivalence identifying, for each $N>1$ and for each $R\in\mathcal{P}_N(\zeta)$, the point $x\in V'(s_N(R))$ to $i_N(x)\in V(R)$.

With this definition the space $\Omega(\zeta)$ has a natural structure as a Riemann surface with boundary. The boundary is the outer boundary of $B_0(R_1)$.

7.2. The potential function H, critical graphs and arguments

Any parapattern naturally comes with a harmonic mapping $H=H(\zeta)\colon\Omega(\zeta)\to\mathbf{R}_+$ with $H(\zeta)|_{B_{N-1}(R)}=h_R|_{B_{N-1}(R)}$.

The *ζ-parapattern of depth $N\geq 0$* is the subset

$$\Omega_N(\zeta)=\left\{(\zeta,x)\in\Omega(\zeta)\,\middle|\,\frac{\log|\zeta|}{3^N}<H(\zeta,x)\leq 3\log|\zeta|\right\}.$$

The critical values of H are the numbers $\log|\zeta|/3^n$, for $n\geq 0$. Let

$$\Gamma_N(\zeta)=\left\{(\zeta,x)\in\Omega(\zeta)\,\middle|\,H(\zeta,x)=\frac{\log\zeta}{3^{n-1}}\right\}\quad\text{for}\quad n\geq 0.$$

Arguments of points $(\zeta,x)\in\Omega(\zeta)$ and arguments of graphs are defined analogously to arguments of points in patterns in Section 2.4.

7.3. The real parapattern

Choose $r>1$ real. Figure 7.1 represents $\Omega_3(r)$ (cf. Figure 2.5 of the tree of real patterns down to level 3).

The following proposition is the analogue of Proposition 2.6.

PROPOSITION 7.1. *Each critical point x of the potential function $H=H(r)$: $\Omega(r)\to\mathbf{R}_+$ satisfies $H(x)=\log r/3^{k-1}$ for some $k\geq 1$. It has precisely two ascending rays,*

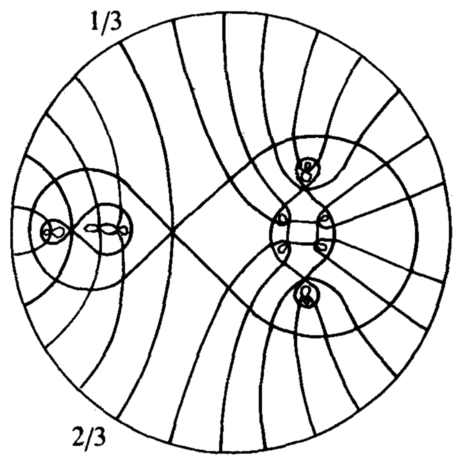


Fig. 7.1

which intersect the boundary $\Gamma_0(r)$ of $\Omega(r)$ at points with arguments of the form $p/3^k$ for some p not divisible by 3. *Moreover all such rational arguments are obtained this way.*

Proof. The proof of course goes by induction. Start the induction by observing that there are precisely two ascending rays emanating from the unique critical point of H in $\Gamma_1(r)$ with arguments 1/3 and 2/3.

The inductive hypothesis says that the result is true for all $k<N$. Take any argument of the form $p/3^N$ with $(p,3)=1$; using Proposition 2.6 we get a unique pattern $R\in\mathcal{P}_N(r)$ for which $p/3^N\in\arg(\Gamma'_N(R))$. The descending ray of that argument leads to a critical point $x\in\Gamma'_N(R)$, from which exactly one other ascending ray emanates, and leads to a point of argument $p'/3^N$ for some p' not divisible by 3. The point x belongs to $V(R)$, which is contained in $\Omega(r)$. Q.E.D. for Proposition 7.1

7.4. Parapattern isomorphisms

Let $T(\zeta)\colon \Omega(\zeta)\to\Omega(-\zeta)$ be the mapping defined by

$$T(\zeta)|_{V(R)} = \tau_R|_{V(R)}$$

where τ_R is the involution from Proposition 2.9. This mapping T is well defined because τ_R is compatible with i_R.

PROPOSITION 7.2. *The mapping* $T(\zeta)\colon \Omega(\zeta)\to\Omega(-\zeta)$ *is an analytic isomorphism. If* $F\colon \Omega(\zeta)\to\Omega(\eta)$ *is an analytic isomorphism, then either* $\eta=\zeta$ *and* $F=\mathrm{id}$ *or* $\eta=-\zeta$ *and* $F=T$.

Proof. The mapping T is analytic because its restriction to each $V(R)$ is analytic.

The potential H on a parapattern can be reconstructed as follows. Suppose W is a parapattern. Consider the unique harmonic function h on W taking the value 1 on the (outer) boundary of W and the value 0 at the ends. This function differs (a posteriori) from H by an multiplicative constant. The constant in question can be determined from the moduli M_1 and M_2 of the two annuli between $h^{-1}(a)$ and $h^{-1}(b)$, where a and b are the two highest critical values of h. It will always be true (still a posteriori) that up to numbering $M_1=2M_2$, and then

$$H=3\pi M_1 h.$$

We know $|\zeta^2|$ since $\log|\zeta^2|=2\pi M_1$. We can reconstruct the argument of ζ^2 exactly as we did for a pattern in Remark 2.11. So W equals either $\Omega(\zeta)$ or $\Omega(-\zeta)$ for some ζ and it is not possible to distinguish them, since they are analytically isomorphic as explained above. Q.E.D. for Proposition 7.2

7.5. The parapattern bundle

The entire construction of a parapattern is functorial, which means that we can put parameters into the construction.

THEOREM 7.3. (a) *There exists a unique topology on* $\tilde{\Omega}=\cup\Omega(\zeta)$ *making it a manifold with boundary and a unique structure of a 2-dimensional complex manifold on the interior such that the canonical map* $V_N\to\tilde{\Omega}$ *is an analytic chart for every* $N\geq 1$.

(b) *With this structure, the projection*

$$\tilde{p}\colon \tilde{\Omega}\to \mathbf{C}\text{–}\bar{D}$$

given by $\tilde{p}(\zeta,x)=\zeta$ *for* $(\zeta,x)\in\mathrm{int}(\Omega(\zeta))$ *is an analytic submersion.*

Proof. The canonical maps $V_N\to\tilde{\Omega}$ are injective: the equivalence relation never identifies different points of U_N. The charts above clearly cover the interior of $\tilde{\Omega}$. Since the equivalence is induced by the maps i_N which are analytic by Proposition 2.13, these charts define a complex structure. Q.E.D. for Theorem 7.3

7.6. The quotient parapattern bundle

PROPOSITION 7.4. *The isomorphisms $T(\zeta)\colon \Omega(\zeta)\to\Omega(-\zeta)$ defined in Proposition 7.2 define an analytic involution of the bundle $\tilde{\Omega}$ covering the involution $\zeta\mapsto-\zeta$ of $\mathbf{C}-\bar{D}$.*

The proof is left to the reader.

Define the analytic manifold with boundary $\bar{\Omega}=\tilde{\Omega}/T$; the mapping $\bar{p}([z])=(p(z))^2$ is an analytic submersion.

7.7. The bundle of clover leaves

We define an equivalence relation on $\bar{\Omega}$ as follows:

$(\zeta_1,x)\in\Omega(\zeta_1)$ and $(\zeta_2,y)\in\Omega(\zeta_2)$ are equivalent if and only if $(\zeta_1,x)=(\zeta_2,y)$ or $\zeta_1^{\,3}=\zeta_2^{\,3}$ and $x=\zeta_1^{\,3}$, $y=\zeta_2^{\,3}$.

The quotient space

$$\Omega=\bar{\Omega}/\sim$$

with the quotient topology is a manifold with a "ramified boundary" $\partial\Omega=\partial\bar{\Omega}/\sim$ (see the remark after Proposition 12.9 in [BH]).

This requires a bit of explanation: the boundary of $\Omega(\zeta)$ is the circle $z=|\zeta|^3$, so (ζ,ζ^3) is a point of the boundary of $\Omega(\zeta)$. Thus the only gluing taking place is to identify the point $(\zeta,\zeta^3)\in\partial(\Omega(\zeta))$ to the points $(j\zeta,\zeta^3)\in\partial(\Omega(j\zeta))$ and $(j^2\zeta,\zeta^3)\in\partial(\Omega(j^2\zeta))$, where $j=e^{2\pi i/3}$. In particular the map

$$p\colon\Omega\to\mathbf{C}-\bar{D}$$

given by $(\zeta,z)\mapsto\zeta^3$ is well defined, and its fiber above ζ^3 is $\Omega(\zeta)\cup\Omega(j\zeta)\cup\Omega(j^2\zeta)$, glued to each other by identifying one point of each boundary. (This is closely related to the content of Remark 3.4.)

Chapter 8. Parapatterns and polynomials

In this chapter we shall show that every pattern of finite depth can be realized by a polynomial. This will allow us to identify the *escape locus*, i.e. the part of parameter space for cubic polynomials in which both critical points escape, with the parapattern bundle.

8.1. The universality of parapatterns

Now suppose that Λ is an analytic manifold, and that $(P_\lambda)_{\lambda\in\Lambda}$ is a family of monic cubic polynomials depending analytically on Λ and with labelled critical points $\omega_1(\lambda)$, $\omega_2(\lambda)$ being analytic functions of the parameter.

If for all $\lambda\in\Lambda$, we have $h_\lambda(\omega_1(\lambda))>h_\lambda(\omega_2(\lambda))>0$, then by Corollary 3.3 there exists a unique $\zeta(\lambda)$, a unique $N(\lambda)$, a unique pattern $R(\lambda)\in\mathcal{P}_{N(\lambda)+1}(\zeta(\lambda))$ and an isomorphism $\tilde{\varphi}_\lambda\colon U_\lambda(\omega_2(\lambda))\to R(\lambda)$ which is defined at the critical value $P_\lambda(\omega_2(\lambda))$. Recall from Section 2.8 the subsets $V'(R(\lambda))\subset V(R(\lambda))\subset R(\lambda)$. Clearly

$$\Phi_\Lambda(\lambda)=\tilde{\varphi}_\lambda(P_\lambda(\omega_2(\lambda)))$$

is in the $V(R(\lambda))-V'(R(\lambda))$.

Using the Remark 3.4, the construction above still goes through if $h_\lambda(\omega_1(\lambda))=h_\lambda(\omega_2(\lambda))$ and $\omega_1(\lambda)\neq\omega_2(\lambda)$, and in case $\omega_1(\lambda)=\omega_2(\lambda)$, the identification we have made on $\bar{\Omega}$ still makes the mapping $\Phi_\Lambda\colon\Lambda\to\Omega$ well defined.

PROPOSITION 8.1. *Let Λ be an analytic manifold with boundary and $(P_\lambda)_{\lambda\in\Lambda}$ an analytic family of monic cubic polynomials with labelled critical points $\omega_1(\lambda)$, $\omega_2(\lambda)$ depending analytically on $\lambda\in\mathrm{int}(\Lambda)$, and satisfying $h_\lambda(\omega_1)\geq h_\lambda(\omega_2)>0$. Then the mapping $\Phi_\Lambda\colon\Lambda\to\Omega$ is continuous and analytic on* int Λ.

Proof. If the family $(P_\lambda)_{\lambda\in\Lambda}$ is constant, the result is obvious. Otherwise, if $\lambda_0\in\mathrm{int}(\Lambda)$, then $\Phi_\Lambda(\lambda_0)\in\mathrm{int}\,\Omega$ since $h_\lambda(\omega_1)-h_\lambda(\omega_2)$ is harmonic on $\mathrm{int}(\Lambda)$. Set $\mathrm{depth}(P_{\lambda_0})=N(\lambda_0)=N$. There exists a neighborhood $\Lambda'\subset\Lambda$ of λ_0 such that, for all $\lambda\in\Lambda'$, we have $N\leq N(\lambda)\leq N+1$ and $\varphi_\lambda(P_\lambda(\omega_2(\lambda)))$ lies in V_{N+1}; since V_{N+1} is a chart of Ω it follows that the mapping φ_Λ is continuous at λ_0 and analytic in a neighborhood of λ_0 if $\lambda_0\in\mathrm{int}\,\Lambda$.

If $\lambda_0\in\partial\Lambda$ and $\Phi_\Lambda(\lambda_0)\in\partial\Omega$ then the two critical points $\omega_1(\lambda_0)$, $\omega_2(\lambda_0)$ escape at the same rate. There exists a neighborhood $\Lambda'\subset\Lambda$ of λ_0 such that, for all $\lambda\in\Lambda'$, we have $N(\lambda)=0$. In this case $P_\lambda(\omega_2(\lambda))$ is in the domain of φ_λ. Q.E.D. for Proposition 8.1

8.2. Parapatterns and the escape locus

Now let us apply Proposition 8.1 to a particular Λ. But first we need some notation.

We shall parametrize cubic polynomials by

$$P_{a,b}(z)=z^3-3a^2z+b.$$

The critical points of $P_{a,b}$ are $\pm a$. Let $\mathcal{S}$ be the set of parameters for which at least one critical point escapes to infinity; i.e.

$$\mathcal{S}=\{(a,b)\in \mathbf{C}^2 \mid \sup(h_{a,b}(-a), h_{a,b}(a))>0\}.$$

This set naturally splits into two symmetrical sets $\mathcal{S}=\mathcal{S}^+\cup\mathcal{S}^-$ where

$$\mathcal{S}^+=\{(a,b)\in\mathcal{S} \mid 0<h_{a,b}(-a)\leq h_{a,b}(a)\}.$$

The labelled critical points of $P_{a,b}$ for $(a,b)\in\mathcal{S}^+$ are $\omega_1(a,b)=a$ and $\omega_2(a,b)=-a$, and the co-critical point $\omega_1'(a,b)$ is $-2a$.

Proposition 11.3 and Corollary 13.3 in [BH] state that the mapping $\psi^+ \colon \mathcal{S}^+\to\mathbf{C}\text{–}\bar{D}$ given by $\psi^+(a,b)=\varphi_{a,b}(P_{a,b}(a))$ is a non-trivial fiber bundle with fibers homeomorphic to a trefoil clover leaf, i.e. three closed discs with one boundary point in common. Moreover this corollary implies that the mapping

$$\tilde{\psi}^+\colon \mathcal{S}^+-\partial\mathcal{S}^+\to C\text{–}\bar{D}$$

given by $\tilde{\psi}^+(a,b)=\lim_{z\to-2a}\varphi_{a,b}(z)$ is a non trivial fiber bundle with fibers homeomorphic to D. Note that $-2a$ is the co-critical point of a so that $\tilde{\psi}^+$ is a cube root of ψ^+ which is well defined as long as the locus where the critical points coincide has been removed from the domain. (See Remark 3.4.) We shall denote the fiber above ζ by

$$\mathcal{L}^+(\zeta)=(\tilde{\psi}^+)^{-1}(\zeta),$$

the *ζ-leaf* of the trefoil clover.

In this section we shall only consider a subset of $\mathcal{S}$: the *escape locus* $\mathcal{E}$ which is the set of parameters for which both critical points escape to infinity; i.e.

$$\mathcal{E}=\{(a,b) \mid \inf(h_{a,b}(-a), h_{a,b}(a))>0\}.$$

Let $\mathcal{E}^+=\mathcal{S}^+\cap\mathcal{E}$ and $\mathcal{E}^+(\zeta)=\mathcal{L}^+(\zeta)\cap\mathcal{E}$. In Chapter 10 we shall consider the complementary subset $\mathcal{B}=\mathcal{S}\text{–}\mathcal{E}$, i.e., the set of parameters for which one critical point escapes to infinity and the other critical point has bounded orbit. Let $\mathcal{B}^+=\mathcal{S}^+\cap\mathcal{B}$ and $\mathcal{B}^+(\zeta)=\mathcal{L}^+(\zeta)\cap\mathcal{B}$. We have $\mathcal{S}=\mathcal{E}\cup\mathcal{B}$, $\mathcal{S}^+=\mathcal{E}^+\cup\mathcal{B}^+$ and $\mathcal{L}^+(\zeta)=\mathcal{E}^+(\zeta)\cup\mathcal{B}^+(\zeta)$.

Now we are ready to apply Proposition 8.1 to $\Lambda=\mathcal{E}^+$.

THEOREM 8.2. *The mapping $\Phi_{\mathcal{E}^+}\colon \mathcal{E}^+\to\Omega$ is a homeomorphism, analytic in the interior.*

Proof. We have seen in Proposition 8.1 that $\Phi_{\mathscr{C}^+}$ is analytic in the interior of $\mathscr{C}^+-\partial\mathscr{C}^+$; we will now show that it is proper of degree one. Let $P_n\in\mathscr{C}^+$ be a sequence of polynomials with the sequence $z_n=\Phi_{\mathscr{C}^+}(P_n)$ convergent in Ω. By using the local triviality of the bundle $\psi^+:\mathscr{S}^+\to\mathbf{C}-\bar{D}$ and the compactness of each fiber we can assume that the P_n converge to a polynomial P. Since the sequence (z_n) is convergent, the sequence lies in Ω_N for some fixed N, so we have the bounds

$$\frac{1}{3^N}\leqslant\frac{h_{a,b}(-a)}{h_{a,b}(a)}\leqslant 1,$$

which are still true of P; so P is in $\mathscr{C}^+$. This shows that the map is proper, hence it has a degree. We showed in [BH], Theorem 12.2 that this degree is 1.

Q.E.D. for Theorem 8.2

Remark 8.3. Theorem 8.2 says that given $R\in\mathscr{P}_N(\zeta)$ and $x\in V(R)-V'(R)$ there exists a unique polynomial $P\in\mathscr{C}^+$ so that $\bar{\varphi}_P(P(-a))=x$. This shows that every ζ-pattern can be realized by a polynomial. Moreover, it identifies the analytic structure of $\operatorname{int}\mathscr{C}^+$ and that of $\operatorname{int}\Omega$. A more precise statement is found in Corollary 8.4 below.

Since the mapping $\bar{p}:\bar{\Omega}\to\mathbf{C}-\bar{D}$ is not well defined at the equivalence classes we define the mapping $p:\Omega\to\mathbf{C}-\bar{D}$ to be $p(x)=(\bar{p}(x))^3$ if $x\in\operatorname{Int}(\Omega)$ and $p(x)=\zeta^3$ if $x\in\partial\Omega(\zeta)$. This is well defined (even at the gluing point).

COROLLARY 8.4. (a) *The mapping*

$$\begin{tikzcd}
\mathscr{C}^+ \arrow[rr, "\Phi_{\mathscr{C}^+}"] \arrow[dr, "\psi^+|_{\mathscr{C}^+}"'] & & \Omega \arrow[dl, "p"] \\
& \mathbf{C}-\bar{D} &
\end{tikzcd}$$

is a bundle isomorphism.

(b) *The mapping* $(\Phi_{\mathscr{C}^+}|_{\mathscr{C}^+(\zeta)})^{-1}:\Omega(\zeta)\to\mathscr{C}^+(\zeta)$ *is analytic in the interior.*

Remark 8.5. It follows that the mapping $p:\Omega\to\mathbf{C}-\bar{D}$ is a non trivial fiber bundle. Moreover we see that any parapattern $\Omega(\zeta)$ is realized in the parameter space of cubic polynomials.

Chapter 9. Ends of parapatterns and polynomials of infinite depth

The parapattern space $\Omega(\zeta)$ has ends, $E(\Omega(\zeta))$, which correspond bijectively to the connected components of

$$\mathscr{B}^+(\zeta)=\mathscr{L}^+(\zeta)-\mathscr{E}^+(\zeta).$$

The set of connected components of $\mathscr{B}^+(\zeta)$ with the quotient topology is a Cantor set homeomorphic to $E(\Omega(\zeta))$.

We will show in this chapter that countably many of these components are homeomorphic to the Mandelbrot set M and that all other components are points; the dichotomy will correspond precisely to the dichotomy in $\mathscr{P}_\infty(\zeta)$ between patterns with periodic critical ends and non-periodic critical ends which we encountered in Chapters 4 and 5.

9.1. The two types of ends

The appearance of Mandelbrot sets in the parameter space for cubic polynomials is explained by families of polynomial-like mappings of degree 2. We saw in Chapter 5 that some restrictions of iterates of cubic polynomials are polynomial-like of degree 2, and in this section we will consider the subsets of parameter space in which this occurs. We will see that this gives rise to infinitely many Mandelbrot-like families.

Each end of $x\in E(\Omega(\zeta))$ corresponds to a pattern

$$\operatorname{pat}(x)=R\in\mathscr{P}_\infty(\zeta).$$

Indeed, such an end corresponds to a sequence of patterns $(R_0,R_1,R_2,\dots)$ and the sequence $(B_0(R_1),B_1(R_2),\dots)$ of critical value annuli. Conversely, to any pattern $R=(R_0,R_1,R_2,\dots)$ in $\mathscr{P}_\infty(\zeta)$, we can associate the sequence of nested annuli $(B_0(R_1),B_1(R_2),\dots)$, which surrounds a unique end in $E(\Omega(\zeta))$. So $\operatorname{pat}:E(\Omega(\zeta))\to\mathscr{P}_\infty(\zeta)$ establishes a one-to-one correspondence between $E(\Omega(\zeta))$ and $\mathscr{P}_\infty(\zeta)$.

THEOREM 9.1. (a) *If the critical end of* $\operatorname{pat}(x)$ *is periodic then the component M_x of $\mathscr{B}^+(\zeta)=\mathscr{L}^+(\zeta)-\mathscr{E}^+(\zeta)$ corresponding to x is homeomorphic to the Mandelbrot set M.*

(b) *If the critical end of* $\operatorname{pat}(x)$ *is not periodic then the component of $\mathscr{B}^+(\zeta)=\mathscr{L}^+(\zeta)-\mathscr{E}^+(\zeta)$ corresponding to x is a point.*

Proof of (b). Let us first prove (b), which is easy with the main result of Chapter 4. The annuli of the nest of an end x are precisely the same as the annuli of the nest of the critical value end of $R=\mathrm{pat}(x)$. If the critical end of R is not periodic, then the modulus of this end is infinite (Theorem 4.3), so the modulus of the end x is infinite, and the component of $\mathscr{L}^+(\zeta)-\mathscr{C}^+(\zeta)$ corresponding to x must be a point (Propositions 5.4 and 5.5). Q.E.D. for Theorem 9.1 (b)

9.2. Mandelbrot-like families in cubic polynomials

We shall first recall the notion of Mandelbrot-like families of polynomial-like mappings [DH2] since that is needed for the proof of (a).

Let $f=(f_\lambda\colon U'_\lambda\to U_\lambda)_{\lambda\in\Lambda}$ be an analytic family of polynomial-like mappings of degree 2 with Λ homeomorphic to a disc. Let ω_λ denote the critical point of f_λ. Set

$$M_{\mathbf{f}}=\{\lambda\in\Lambda\,|\,K_{\mathbf{f}_\lambda}\text{ connected}\}.$$

It follows from the straightening theorem, quoted in Section 5.1, that there is a map

$$\chi\colon M_{\mathbf{f}}\to M$$

defined by f_λ, $\lambda\in M_{\mathbf{f}}$, is hybrid equivalent to $Q_{\chi(\lambda)}(z)=z^2+\chi(\lambda)$. The family $\mathbf{f}=(f_\lambda)_{\lambda\in\Lambda}$ is called *Mandelbrot-like* if χ is a homeomorphism.

We shall need Proposition 21 from [DH2] in the following form:

PROPOSITION. *Let $\mathbf{f}=(f_\lambda\colon U'_\lambda\to U_\lambda)_{\lambda\in\Lambda}$ be an analytic family of polynomial-like mappings of degree 2 with Λ homeomorphic to D. Suppose there exists a subset $A\subset\Lambda$ homeomorphic to $\bar D$ such that $f_\lambda(\omega_\lambda)\in U_\lambda-U'_\lambda$ for $\lambda\in\Lambda-\mathrm{int}(A)$. The map $\chi\colon M_{\mathbf{f}}\to M$ is a homeomorphism if $f_\lambda(\omega_\lambda)-\omega_\lambda$ turns around 0 once as λ describes ∂A.*

Now suppose $x\in E(\Omega(\zeta))$, $R=(R_0,R_1,R_2,\ldots)=\mathrm{pat}(x)$ and the critical end c of R is periodic of some period k and some level n_0 (cf. Section 5.1)

As recalled in Chapter 8 it follows from Theorem 13.2 and Corollary 13.3 in [BH] that $\mathscr{L}^+(\zeta)$ is homeomorphic to a disc. The critical value annulus $B_{n_0-1}(R)$ is naturally embedded in $\Omega(\zeta)$ and the outer boundary of $\Phi^{-1}_{\mathscr{C}^+}(B_{n_0-1}(R))$ bounds a region Λ_x homeomorphic to an open disc. All the polynomials which belong to Λ_x have patterns which to depth n_0 coincide with R_{n_0}. The corresponding regions of the dynamical planes can therefore be identified through $\bar\varphi_{a,b}$ (Theorem 3.1).

Example 9.2. For each ζ there are unique patterns $R_1,R_2\in\mathscr{P}_\infty(\zeta)$ for which the critical ends c_1,c_2 have period 1 and 2 respectively. The critical value end is $b_i=\pi_{R_i}(c_i)$

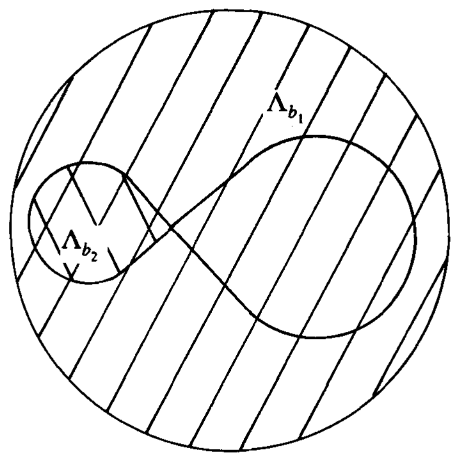


Fig. 9.1

for $i=1,2$. Figure 9.1 shows the regions Λ_{b_1} and Λ_{b_2} corresponding to these ends. Figure 5.4 illustrates any polynomial in $\Lambda_{b_1}=\mathscr{L}^+(\zeta)$; these are the "polynomial-like mappings which stand out particularly" mentioned in the introduction to Chapter 5. Figure 5.6 illustrates any polynomial in Λ_{b_2}.

Recall from Section 5.1 the definition of W_n as the smallest simply-connected domain in $\mathbf{C}$ containing the critical annulus $C_n(P)$. In Chapter 5 they are only considered for polynomials of infinite depth, but the definition clearly makes sense even if P has finite depth so long as $n \geq \text{depth}(P)$.

PROPOSITION 9.3. *Let $x \in E(\Omega(\zeta))$. Suppose the critical end of* pat(x) *is periodic of period k and level n_0.*

(a) *For all $(a,b) \in \Lambda_x$, the restriction of the k-th iterate of $P_{a,b}$ to W_{a,b,n_0}:*

$$f_{a,b} = P_{a,b}^{\circ k}|_{W_{a,b,n_0}} : W_{a,b,n_0} \to W_{a,b,(n_0-k)}$$

is polynomial-like of degree 2, where $W_{a,b,n}$ is the smallest simply-connected domain in $\mathbf{C}$ containing $C_n(P_{a,b})$.

(b) *A polynomial $P_{a,b}$ with $(a,b) \in \Lambda_x$ belongs to M_x if and only if the critical point of the polynomial-like mapping $f_{a,b}$ does not escape.*

(c) *The component M_x is homeomorphic to the Mandelbrot set M.*

Proof. (a) and (b) The polynomials $P_{a,b}$ for $(a,b) \in \Lambda_x$ satisfy the requirements of Proposition 5.7.

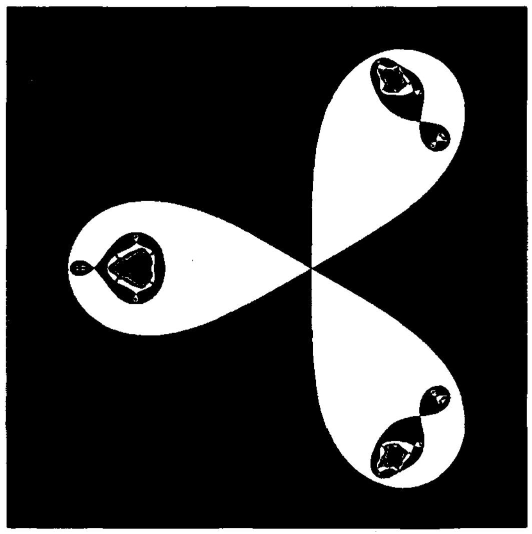


Fig. 9.2

(c) This follows from Proposition 21 in [DH2] stated above. Choose any ϱ with

$$\frac{\log|\zeta|}{3^{n_0}}<\varrho<\frac{\log|\zeta|}{3^{n_0-1}}.$$

Let

$$A_x=\{(a,b)\in\Lambda_x|H\circ\Phi_{g^+}(a,b)\leq\varrho\},\quad \gamma=\partial A_x \quad\text{and}\quad \delta=\pi_R^{\circ(k-1)}(\Phi_{g^+}(\gamma)).$$

Note that the last formula above makes sense because $\Phi_{g^+}(\gamma)$ lies in the annulus $B_{n_0-1}(R)$ which is naturally a subset of both R and $\Omega(\zeta)$.

When (a,b) describes the boundary curve γ, turning around M_x once, then $\bar{\varphi}_{a,b}(P_{a,b}(-a))$ describes the curve $\Phi_{g^+}(\gamma)$ in $B_{n_0-1}(R)\subset R$. Since $\pi_R^{\circ(k-1)}$ maps $B_{n_0-1}(R)$ with degree 1 onto $C_{n_0-k}(R)$ then $\bar{\varphi}_{a,b}(P_{a,b}^{\circ k}(-a))$ describes δ turning around the critical end c once. Hence the vector from $-a$ to $P_{a,b}^{\circ k}(-a)$ turns around 0 once. We have exactly verified the hypotheses of the proposition above. Q.E.D. for Proposition 9.3

This ends the proof of (a) in Theorem 9.1. Q.E.D. for Theorem 9.1

Remark 9.4. Theorem 9.1 (b) says that given a pattern $R\in\mathcal{P}_\infty(\zeta)$ with non-periodic critical end there exists a unique polynomial of infinite depth realizing R. Moreover, Theorem 9.1 (a) says that given a pattern $R\in\mathcal{P}_\infty(\zeta)$ with periodic critical end there exist infinitely many polynomials of infinite depth realizing R.

(a) (b)

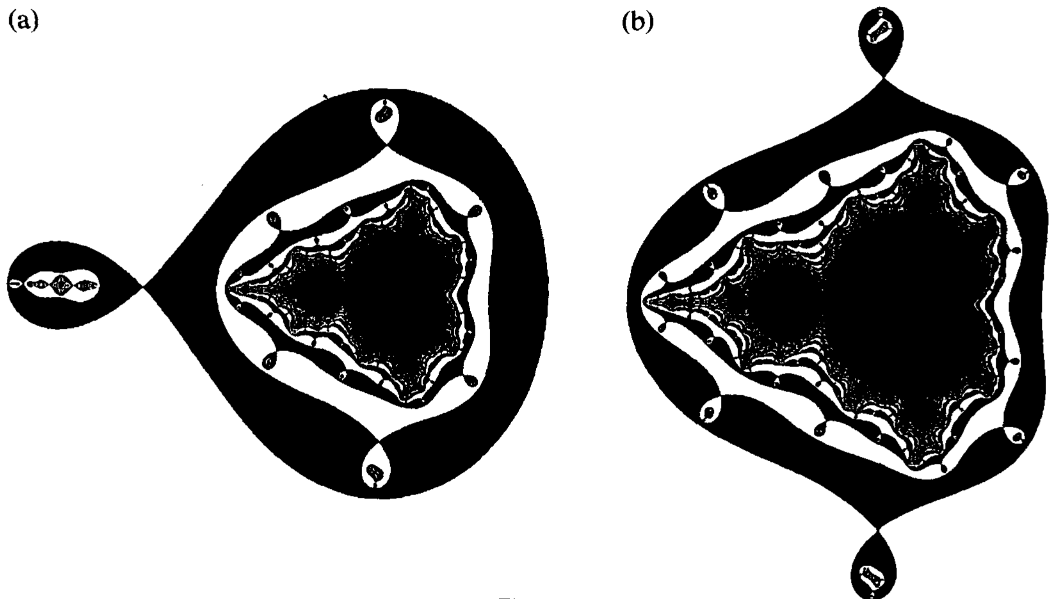


Fig. 9.3

Recall from Section 8.2 that $\psi^+: \mathscr{S}^+ \to \mathbf{C}-\bar{D}$ given by $\psi^+(a,b)=\varphi_{a,b}(P_{a,b}(a))$ is a non-trivial fiber bundle with fibers homeomorphic to a trefoil clover leaf. Figure 9.2 shows the fiber above r for some $r>1$.

Figure 9.3 (a) shows a part of $\mathscr{L}^+(r)$ and Figure 9.3 (b) a blow up around M_{b_1} in Λ_{b_1}.

Figures 9.4 (a) and (b) show two successive blow ups of $\mathscr{L}^+(r)$ around M_{b_2} in Λ_{b_2}.

(a) (b)

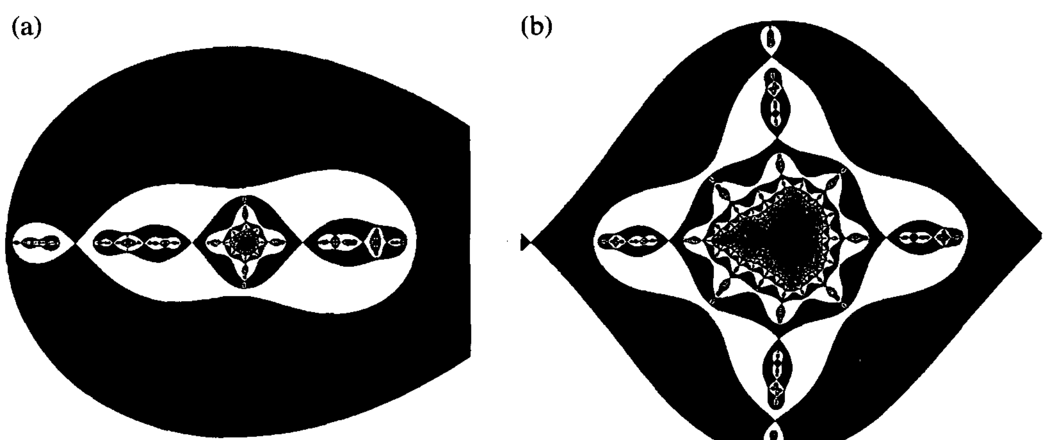


Fig. 9.4

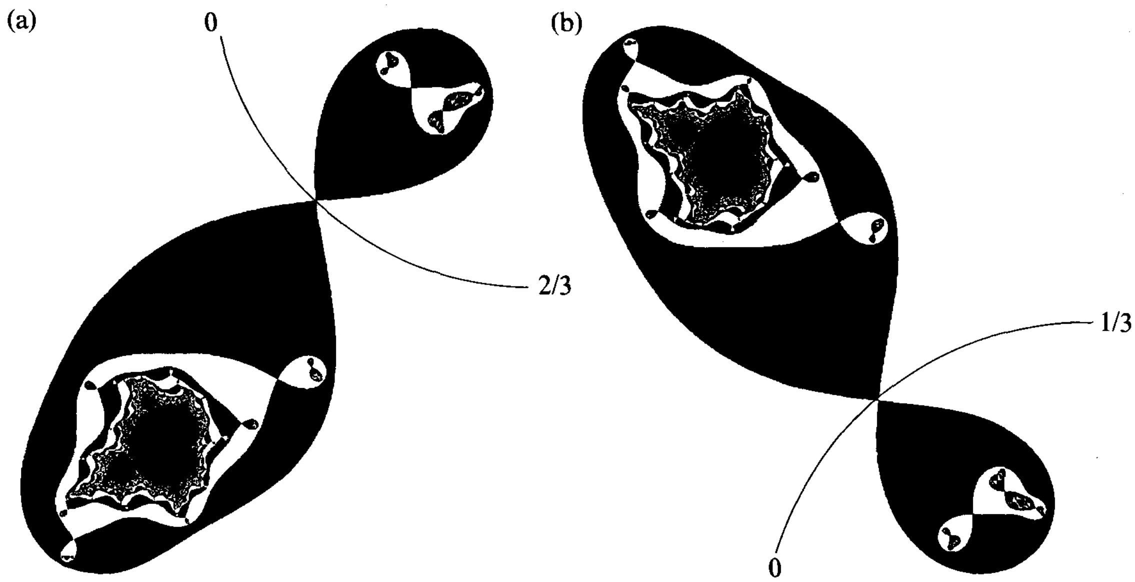


Fig. 9.5

Figures 9.5 (a) and (b) show parts of $\mathscr{L}^+(re^{2\pi i/3})$ and $\mathscr{L}^+(re^{4\pi i/3})$ respectively.

The Julia sets shown in Figures 5.7, 5.8, 5.9 correspond to the polynomials chosen to be the center of the Mandelbrot set M_{b_1} in Figures 9.3, 9.5 (a) and (b) respectively. The Julia set shown in Figure 5.10 corresponds to the polynomial at the center of the hyperbolic component of period 2 of the Mandelbrot set M_{b_1} in $\mathscr{L}^+(r)$. The Julia set shown in Figure 5.11 corresponds to the polynomial at the center of the Mandelbrot set M_{b_2} in $\mathscr{L}^+(r)$. The Julia set shown in Figure 5.12 corresponds to the polynomial at the tip of the Mandelbrot set M_{b_1} in $\mathscr{L}^+(r)$.

Chapter 10. The monodromy of parapatterns

Recall from Section 8.2 that $\psi^+: \mathscr{S}^+ \to \mathbf{C} - \bar{D}$ given by $\psi^+(a, b) = \varphi_{a,b}(P_{a,b}(a))$ is a non trivial fiber bundle with fibers homeomorphic to a trefoil clover leaf.

In Corollary 8.4 we have proved that $\Phi_{\mathscr{E}^+}: \mathscr{E}^+ \to \Omega$ is a bundle isomorphism.

In this chapter we shall describe the non-triviality of the parapattern bundle $\tilde{p}: \tilde{\Omega} \to \mathbf{C} - \bar{D}$ and the parapattern quotient bundle $\bar{p}: \bar{\Omega} \to \mathbf{C} - \bar{D}$ by describing their monodromies. All the work was done in Chapter 6; here we just collect results.

Moreover, the monodromies of the parapattern bundle and the quotient parapattern bundle induce monodromies of the ends. We shall use this in Section 10.5 to describe the components of the subset $\mathscr{B}$ of parameter space for cubic polynomials with one critical point escaping to infinity and the other not.

10.1. The local trivialization of the parapattern bundle

The local trivializations

$$\begin{tikzcd} U_N(\zeta)\times U \arrow[r, "f_{R,u}"] \arrow[d, dash] & p_N^{-1}(V) \arrow[d, dash] \\ U \arrow[r, "u\mapsto \zeta|\zeta|^{u-1}"] & V \end{tikzcd}$$

described in Proposition 6.1 induce local trivializations of the subbundle

$$\begin{tikzcd} V_N \arrow[dr] & \subset & U_N \arrow[dl] \\ & \mathbf{C}-\bar{D}. & \end{tikzcd}$$

Moreover, these local trivializations are compatible with the inclusions: the diagram

$$\begin{tikzcd} & V_N(\zeta)\times U \arrow[rr] \arrow[dd] & & V_N|_V \arrow[dd] \\ V'_{N-1}(\zeta)\times U \arrow[ur, "i_N\times \mathrm{id}"] \arrow[rr] \arrow[dr] & & V'_N|_V \arrow[ur, "i_N"] \arrow[dr] & \\ & U \arrow[rr, "u\mapsto \zeta|\zeta|^{u-1}"] & & V \end{tikzcd}$$

commutes. All of this is tedious to verify but essentially obvious, and left to the long-suffering reader.

As a result, the local trivializations of the pattern bundle given by wringing the complex structure induce local trivializations of the bundle $\bar{p}\colon \bar{\Omega}\to \mathbf{C}-\bar{D}$, and just as in Section 6.2 these local trivializations are canonical and define a global trivialization of the pullback of the bundle to the universal cover of $\mathbf{C}-\bar{D}$.

10.2. The monodromy of the parapattern bundle

As in Chapter 6 we choose some $r>1$ as base point in $\mathbf{C}-\bar{D}$ and describe the monodromy as a homeomorphism $m\colon \Omega(r)\to\Omega(r)$.

Using the notation of Theorem 6.3, the discussion above can be summed up as follows:

THEOREM 10.1. *The monodromy* $m\colon \Omega(r)\to\Omega(r)$ *of the bundle* $\bar{p}\colon \bar{\Omega}\to\mathbf{C}-\bar{D}$ *induced by the canonical local trivialization above is given by*

$$m|_{V(R)} = m_N|_{V(R)}$$

where $R\in\mathcal{P}_N(r)$.

Let A be any annulus at level N of $\Omega(r)$. We define the *period of A under the monodromy m* to be the smallest k such that $m^{\circ k}(A)=A$. Since each nested sequence of annuli in $\Omega(r)$ is the nested sequence of critical value annuli $(B_0(R), B_1(R), \ldots)$ for some $R\in\mathcal{P}_\infty(r)$ we get the following proposition as an immediately consequence of Proposition 6.4.

PROPOSITION 10.2. (a) *The periods under the monodromy m of a nested sequence of annuli* $A_0, A_1, \ldots, A_N, \ldots$ *of* $\Omega(r)$ *form an increasing sequence of powers of* 2.

(b) *Let* $x\in E(\Omega(r))$. *Suppose the critical end of* $R=\mathrm{pat}(x)\in\mathcal{P}_\infty(r)$ *is periodic. Then the periods under the monodromy m of the annuli in the nest of x are bounded.*

10.3. The monodromy of the quotient parapattern bundle

The monodromy of the intermediate bundle

$$\bar{p}\colon \bar{\Omega}\to\mathbf{C}-\bar{D}$$

is a "functional square root" of the mapping m above.

We chose $r>1$ as base point in $\mathbf{C}-\bar{D}$ for the bundle $\bar{p}\colon \bar{\Omega}\to\mathbf{C}-\bar{D}$, so it is natural to choose $r^2\in\mathbf{C}-\bar{D}$ as the base point for $\bar{p}\colon \bar{\Omega}\to\mathbf{C}-\bar{D}$ since $\Omega(r)$ is naturally isomorphic to $\bar{\Omega}(r^2)$.

Using the notation of Theorem 6.5, we have:

THEOREM 10.3. (a) *The monodromy* $\bar{m}\colon \bar{\Omega}(r^2)\to\bar{\Omega}(r^2)$ *of the bundle* $\bar{p}\colon \bar{\Omega}\to\mathbf{C}-\bar{D}$ *is given by*

$$\bar{m}|_{V_N R} = \bar{m}_N(V_N(R))$$

where $R \in \mathcal{P}_N(r)$.

(b) *The monodromy* m *of* $p\colon \Omega(r) \to \mathbf{C} - \bar{D}$ *is given by* $(\bar{m})^{\circ 2}$.

PROPOSITION 10.4. (a) *The periods under the monodromy* $\bar{m}$ *of a nested sequence of annuli* $A_0, A_1, \ldots, A_N, \ldots$ *of* $\bar{\Omega}(r^2)$ *form an increasing sequence of powers of* 2.

(b) *Let* $x \in E(\Omega(r))$. *Suppose the critical end of* $R = \mathrm{pat}(x) \in \mathcal{P}_\infty(r)$ *is periodic. Then the periods under the monodromy* $\bar{m}$ *of the annuli in the nest of* x *are bounded.*

Proofs. Again this is obvious and left to the reader.

10.4. Fractional Dehn twists

As in Chapter 6 it is possible to be more explicit about the monodromy in terms of fractional Dehn twists.

Recall that for each annulus A in $\Omega(r)$ the fraction $q(A)$ is defined by

$$q(A) = 2\pi \frac{\mathrm{mod}(A)}{\log r^2} = \frac{\mathrm{mod}(A)}{\mathrm{mod}(C_0)}.$$

The following theorem is an immediate consequence of Theorem 6.3 and Theorem 10.3.

THEOREM 10.5. *The monodromy* $\bar{m}\colon \bar{\Omega}(r^2) \to \bar{\Omega}(r^2)$ *of the bundle* $\bar{p}\colon \bar{\Omega} \to \mathbf{C} - \bar{D}$ *is given recursively as follows:*

(a) *On* $\partial\Omega(r)$ *the monodromy is the identity.*

(b) *Suppose* $\bar{m}_{N-1}$ *is given by induction. Let* A *be an annulus at level* N *with outer boundary* γ. *Then the mapping* $\bar{m}_N$ *is the extension of* $\bar{m}_{N-1}|_\gamma$ *as a fractional Dehn twist of fraction* $q(A)$.

In Figure 10.1 we label the annuli of $\Omega(r)$ down to level 3 by the fraction of Dehn twist which occurs there. We shall make use of this in Sections 10.5 and 11.4.

10.5. Consequences for the polynomials of infinite depth

The monodromy described for the parapattern bundle induces a monodromy on the ends of $\Omega(r)$, i.e., a homeomorphism

$$m\colon E(\Omega(r)) \to E(\Omega(r)).$$

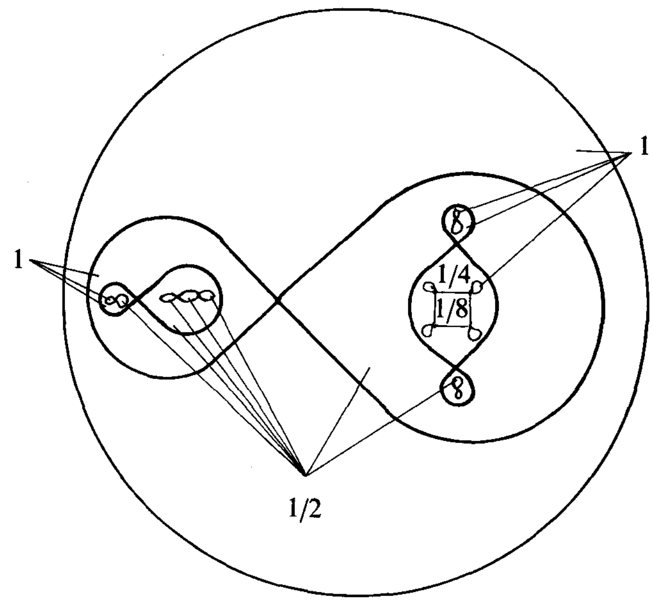


Fig. 10.1

We define a metric on the set of ends $E(\Omega(r))$ as follows:

$$d(x,y)=\frac{1}{3^{n+1}}\quad\text{for}\quad x,y\in E(\Omega(r)),$$

where n is the smallest level at which there exists an annulus surrounding both x and y, or in other words if we have to climb to the critical graph Γ_{n+1} in order to go from x to y.

With this metric the set of ends $E(\Omega(r))$ forms a metric Cantor set and $m\colon E(\Omega(r))\to E(\Omega(r))$ is an isometry.

The *orbit* of an end $x\in E(\Omega(r))$ under m is defined to be $(m^{\circ n}(x))_{n\in\mathbf{Z}}$.

PROPOSITION 10.6. *If E is a metric Cantor set and $m\colon E\to E$ an isometry, then the closure of an orbit is a finite set or a Cantor set.*

Proof. Suppose $X=((m^{\circ n}(x))_{n\in\mathbf{Z}}$ is an infinite orbit. In order to prove that $\bar{X}$ is a Cantor set we only have to prove that X is perfect. If not, there would exist $\varepsilon>0$ and $a\in X$ such that $d(a,x)>\varepsilon$ for all $x\in X-\{a\}$, i.e. $d(a,m^{\circ n}(a))>\varepsilon$ for all $n\neq 0$. Using that m is an isometry we get $d(m^{\circ n}(x), m^{\circ k}(x))>\varepsilon$ for all $n\neq k$ which contradicts the compactness of E. Q.E.D. for Proposition 10.6

An end $x \in E(\Omega(r))$ is convergent if the modulus of the nest $N(x)$ is finite and divergent otherwise. Recall that an end $x \in E(\Omega(r))$ is convergent if and only if $R=\mathrm{pat}(x) \in \mathcal{P}_\infty(r)$ has a critical end which is periodic under π_R. It follows from Proposition 10.2(b) that the period under m of a convergent end x is finite.

An end $x \in E(\Omega(r))$ is therefore either

(1) convergent,

(2) divergent with finite orbit under m or

(3) divergent with infinite orbit under m.

We shall prove that these 3 types of ends of $\Omega(r)$ correspond to 3 different types of connected components of $\mathcal{B}^+=\mathcal{S}^+-\mathcal{C}^+$ in the bundle $\tilde{\psi}^+\colon \mathcal{B}^+ \to \mathbf{C}-\bar{D}$ where $\tilde{\psi}^+(a,b)=\bar{\varphi}_{a,b}(-2a)$ (see Section 8.2). For convenience let us discuss the components of the subbundle

$$\tilde{\psi}_r^+\colon \mathcal{B}_r^+ \to S^1$$

where

$$\mathcal{B}_r^+ = \{(a,b) \in \mathcal{B}^+ \mid h_{a,b}(a) = \log r\} \quad \text{and} \quad \tilde{\psi}_r^+ = \bar{\varphi}_{a,b}(-2a)/r.$$

In order to state the result we need the following definition: the *dyadic solenoid* is the projective limit

$$\Sigma = \varprojlim (S^1, z \mapsto z^2)$$

i.e., Σ is the set of sequences $(\ldots, z_2, z_1, z_0)$ with $z_{i+1}{}^2=z_i$ and $|z_0|=1$, endowed with the product topology.

A projective limit

$$\varprojlim (S^1, q_n)$$

with mappings q_n of the form $q_n(z)=z^{2^{k_n}}$ and infinitely many $k_n>0$ is homeomorphic to the dyadic solenoid.

Proposition 10.7. *Any connected component in $\mathcal{B}_r^+$ is homeomorphic to one of the following 3 types:*

(1) *$S^1 \times M$ with the projection $S^1 \times M \to S^1$ wrapping S^1 onto itself 2^k times for some $k \geq 0$,*

(2) *S^1 with the projection $S^1 \to S^1$ wrapping S^1 onto itself 2^k times for some $k \geq 0$,*

(3) *the dyadic solenoid.*

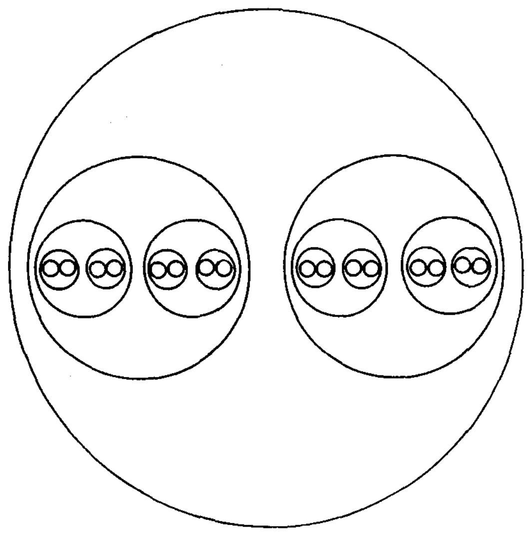


Fig. 10.2

Proof. Type 1 is a component containing $M_x \subset \mathcal{B}^+(r)$ for some convergent end $x \in E(\Omega(r))$. Type 2 is a component containing a point component of $\mathcal{B}^+(r)$ corresponding to a divergent end with finite orbit under m. Type 3 is a component containing a point component of $\mathcal{B}^+(r)$ corresponding to a divergent end with infinite orbit under m. The proof is an immediate consequence of the Propositions 10.2 and 10.6, see also the schematic drawing in Figure 10.2. Q.E.D. for Proposition 10.7

Remark 10.8. (a) In type 1 the bundle $S^1 \times M \to S^1$ can not be like a Möbius band since the Mandelbrot set must be twisted an integer number of times around itself before it closes up. If $x \in E(\Omega(r))$ is convergent and the period of x under m is 2^k then the twisting is determined by $2^k \bmod N(x)$. Hence $2^k \bmod N(x)$ is an integer.

(b) In type 2 and conjecturally in type 1 the connected component is also an arcwise connected component. (Conjecturally refers to the local connectivity conjecture of the Mandelbrot set.) In type 3 the component is not arcwise connected, but the closure of any arcwise connected component.

(c) The dyadic solenoid also occurs in [HO], in the context of complex Hénon mappings.

Let $\bar{\mathcal{B}}^+(\zeta)$ denote the set of equivalence classes in $\mathcal{B}^+(\zeta)$ under affine conjugation and set $\bar{\mathcal{B}}^+ = \cup \bar{\mathcal{B}}^+(\zeta)$. Anything said in this section about components in $\mathcal{B}_r^+$ can be repeated for the bundle $\bar{\mathcal{B}}_r^+ \to S^1$ using the monodromy $\bar{m}$ instead of m.

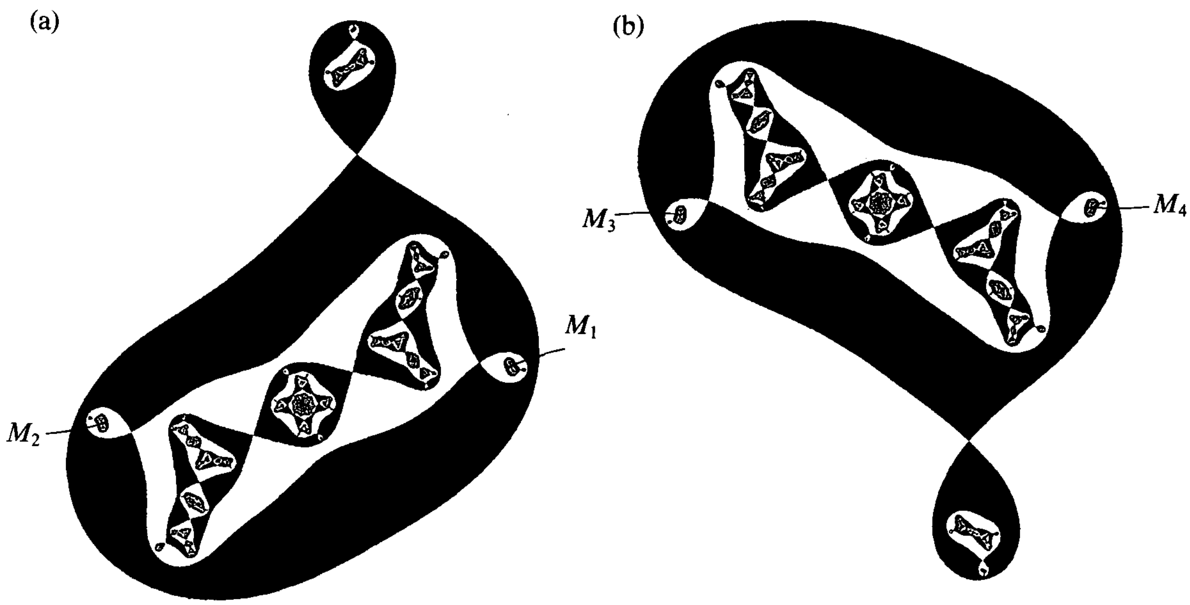


Fig. 10.3

We shall give a sufficient condition for a divergent end x to correspond to a component of type 2.

PROPOSITION 10.9. *Let $x \in E(\Omega(r))$ and $R=\text{pat}(x) \in \mathcal{P}_\infty(r)$. Suppose the critical end of R is non-recurrent. Then x has a finite orbit under $\bar{m}$.*

Proof. Since the critical end of R is non-recurrent there exists an $n \geq 0$ such that any annulus in the nest of x has a modulus equal to $2^k \operatorname{mod}(C_0)$ with $0 \leq k \leq n$. It follows from Theorem 10.5 that the mapping $\bar{m}^{\circ 2^n}$ applied to any annulus in the nest of x is a Dehn twist of integer twist, which maps each annulus in the nest of x to itself. For the induced monodromy on the ends we get $\bar{m}^{\circ 2^n}(x)=x$. Q.E.D. for Proposition 10.9

Remark 10.10. (a) Recall that polynomials with one critical point escaping to infinity and the other critical point strictly preperiodic (for instance Brolin's example) have non-recurrent critical tableaus.

(b) In Example 12.4 we shall define a critical marked grid called the Fibonacci grid with the critical end divergent and recurrent. The ends in $E(\Omega(\zeta))$ which are realizations of the Fibonacci grid correspond to a component in $\mathcal{B}_r^+$ of type 3. We don't know if a critical marked grid with the critical end divergent and recurrent can give rise to a component of type 2.

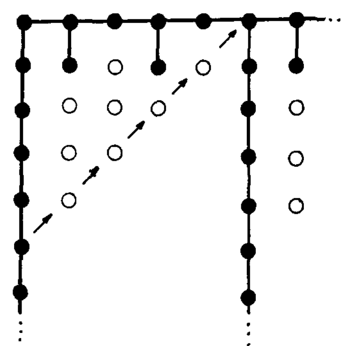


Fig. 10.4

Polynomials belonging to the same component of $\bar{\mathscr{B}}^+$ have patterns which have the same critical marked grids. But there are polynomials not in the same component of $\bar{\mathscr{B}}^+$ which nevertheless have the same marked grid.

Example 10.11. Figure 10.3 (a) and (b) show parts of $\mathscr{L}^+(r)$ for some $r>1$. The figure eights are the parts of $\Gamma_3(r)$ with arguments in [1/9, 2/9] and [7/9, 8/9] respectively. Four Mandelbrot-like sets M_i, $i=1,2,3,4$, are marked. It follows from Theorem 10.5 that M_1 and M_4 are contained in one component of $\bar{\mathscr{B}}^+$ and M_2 and M_3 in another. But they all have the same critical marked grid of period 5 which is shown in Figure 10.4.

Chapter 11. The fundamental group of the escape locus

In this chapter we will compute the fundamental group of the escape locus. This is of interest because [BDK] have proved that the monodromy of Julia sets induces a homomorphism from this fundamental group to the group of automorphisms of the 1-sided 3-shift which is surjective.

There is really only one way to compute fundamental groups: van Kampen's theorem. Our computation is four successive applications of this result. The strategy is as follows.

Recall from Chapter 8 that the escape locus $\mathscr{E}$ naturally splits into two symmetrical sets $\mathscr{E}=\mathscr{E}^+\cup\mathscr{E}^-$ where

$$\mathscr{E}^+=\{(a,b)\,|\,0<h_{a,b}(-a)\leq h_{a,b}(a)\};$$

moreover (Theorem 8.2) the mapping $\Phi_{\mathscr{E}^+}\colon \mathscr{E}^+\to\Omega$ is a homeomorphism. The space Ω is a quotient of $\tilde{\Omega}$ (7.2); and $\tilde{\Omega}$ is a fiber bundle over $\mathbf{C}-\bar{D}$, with fiber say $\Omega(r)$, and monodromy computed in Theorem 10.1.

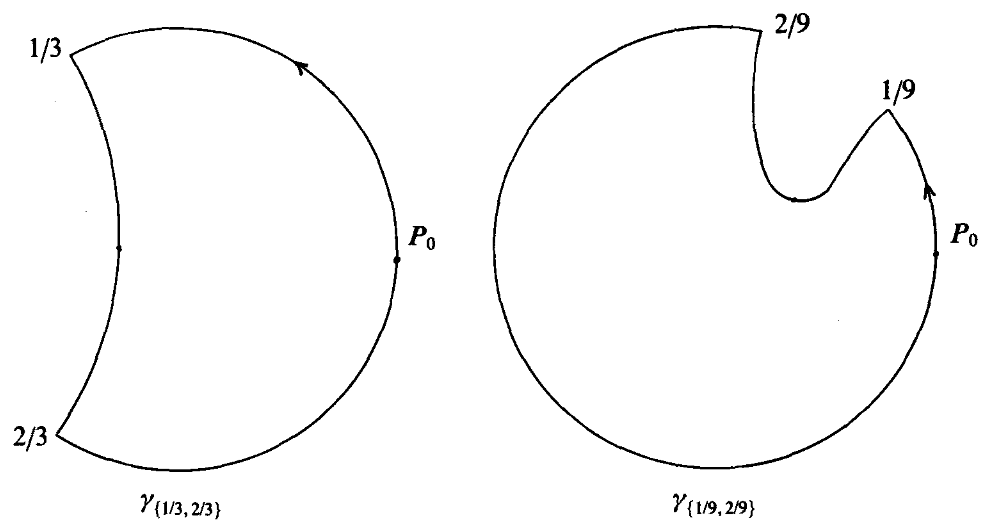


Fig. 11.1

We will use van Kampen's theorem as follows:

(1) Compute the fundamental group of $\Omega(r)$, and the action $\mu:\pi_1(\Omega(r),P_0)\to\pi_1(\Omega(r),P_0)$ of the monodromy on it.

(2) Construct an isomorphism of the fundamental group of $\tilde{\Omega}$ with the semi-direct product $\mathbf{Z}\times_\mu\pi_1(\Omega(r),P_0)$.

(3) The fundamental group of Ω is then given by the presentation $\langle\pi_1(\tilde{\Omega},P_0),\beta\rangle$ where β is a new generator satisfying the relation $\beta^3=\alpha$, where α is a particular element of $\pi_1(\tilde{\Omega},P_0)$, representing the circle which is glued to itself.

(4) Finally, the fundamental group of $\mathscr{E}=\mathscr{E}^+\cup\mathscr{E}^-$ is isomorphic to the amalgamated sum $\pi_1(\mathscr{E}^+,P_0)*_{\pi_1(\mathscr{E}^+\cap\mathscr{E}^-,P_0)}\pi_1(\mathscr{E}^-,P_0)$.

11.1. Choice of base point

First we need to pick a base point. We will choose any polynomial of the form $p_0(z)=z^3+b$ with $b>2/\sqrt{27}$. It is easy to show that such a polynomial belongs to the escape locus, in fact belongs to $\mathscr{E}^+\cap\mathscr{E}^-$. Let $H(P_0)=\log r>0$. The polynomial p_0 doesn't quite have a pattern, since it belongs to the identification locus under the projection $\tilde{\Omega}\to\Omega$; but the point $P_0=(r,r^3)\in\partial\Omega(r)$ is one of the 3 points of $\tilde{\Omega}$ corresponding to this polynomial.

11.2. The fundamental group of the real parapattern

Recall that the lobe $\Omega(r)$, $r>1$, is drawn in Figure 7.1 and is described in Section 7.3.

Proposition 7.1 defines an equivalence relation on $\mathbf{Z}[1/3]/\mathbf{Z}$, with all equivalence classes containing precisely two elements, except the equivalence class of 0 which has only one element. Let X be the set of equivalence classes. For each class $x=\{p_1/3^k, p_2/3^k\}\in X$ we can construct an element

$$\gamma_x \in \pi_1(\Omega(r), P_0)$$

as follows: start at the base point, go counterclockwise around the boundary $\Gamma_0(r)$ until you first meet a point with argument in x, then follow the ray down to the critical point, and go back to $\Gamma_0(r)$ along the other ascending ray leading to the other element of x, then return to the basepoint counterclockwise around the boundary (see Figure 11.1). This definition does not quite work for the element $\{0\}$ of X; the element γ_0 of $\pi_1(\Omega(r), P_0)$ corresponding to $\{0\}$ is simply the boundary of $\Omega(r)$, traversed once counterclockwise.

PROPOSITION 11.1. *The fundamental group $\pi_1(\Omega(r), P_0)$ is the free group on the generators $\gamma_x, x\in X$.*

Proof. This is essentially the easiest case of Morse theory [M1].

First, if X_N is the subset of X with representatives $p/3^k$ with $k\leq N+1$, then γ_x is a loop in $\bar{\Omega}_N(r)$ if $x\in X_N$. It is enough to prove that $\pi_1(\bar{\Omega}_N(r), P_0)$ is the free group on $\gamma_x, x\in X_N$, since the fundamental group of an increasing union is the inductive limit of the fundamental groups; clearly the inductive limit of free groups on increasing finite sets of generators is the free group on the union.

If $N=0$, $\bar{\Omega}_0(r)$ is a deformation retract of the "figure eight" $\Gamma_1(r)$, and $\pi_1(\bar{\Omega}_0(r), P_0)$ is the free group on the two generators γ_0 and $\gamma_{\{1/3,2/3\}}$.

Suppose $N>0$; if y_i are the critical points of H with $H(y_i)=(\log r)/3^N$, then $\bar{\Omega}_{N-1}(r)$ is a deformation retract of $\bar{\Omega}_N(r)-\cup\{y_i\}$. In fact, flow up along the gradient lines of H provides such a retraction, see [M1] for details.

Now apply van Kampen's theorem to the two subsets

$$U_1=\bar{\Omega}_N(r)-\cup\{y_i\} \quad\text{and}\quad U_2=\cup\gamma_i, \quad i\in X_N-X_{N-1},$$

i.e. where the γ_i are those γ_x going through the y_i. The intersection $U_1\cap U_2$ can be deformed to the circle $\Gamma_0(r)$. The fundamental group of $\bar{\Omega}_N(r)$ is the amalgamated sum of the fundamental group of $\bar{\Omega}_{N-1}(r)$, which is by induction the free group on generators

$\gamma_x, x \in X_{N-1}$, and of the fundamental group of $\cup \gamma_i$, which is the free group with generators the γ_i, $i \in X_N - X_{N-1}$ and γ_0, amalgamated over $\pi_1(\Gamma_0(r), P_0)$ which is the free group on γ_0.

The result follows from Lemma 11.2, applied to $Z_1 = (X_N - X_{N-1}) \cup X_0$ and $Z_2 = X_{N-1}$.

Q.E.D. for Proposition 11.1

For any set Z let F_Z be the free group on Z.

LEMMA 11.2. *Let Z_1, Z_2 be two sets; set $Z_0 = Z_1 \cap Z_2$ and $Z = Z_1 \cup Z_2$. Then the diagram*

$$\begin{tikzcd} & F_{Z_1} \arrow[dr] & \\ F_{Z_0} \arrow[ur] \arrow[dr] & & F_Z \\ & F_{Z_2} \arrow[ur] & \end{tikzcd}$$

is an amalgamated sum.

The proof is left to the reader.

11.3. The fundamental group of the parapattern bundle

Let $S^1 = \mathbf{R}/\mathbf{Z}$, $I = [0, 1]$ and $\pi: I \to S^1$ be the canonical map. Let Z be a topological space, and $f: Z \to S^1$ a locally trivial fibration, with pathwise connected fibers. Let $Z_0 = f^{-1}(0)$, and choose a base point $z_0 \in Z_0$.

The pull back of f by π is trivial since I is contractible; a trivialization induces a mapping τ making the diagram

$$\begin{tikzcd} I \times Z_0 \arrow[r, "\tau"] \arrow[d] & Z \arrow[d, "f"] \\ I \arrow[r, "\pi"] & S^1 \end{tikzcd}$$

commute, and which is a homeomorphism on fibers. The induced monodromy $m_\tau: Z_0 \to Z_0$ is given by

$$m_\tau(z) = \tau(1, z).$$

The mapping τ can be chosen so that $m_\tau(z_0)=z_0$. In that case the map $\alpha: I\to Z$ given by $t\mapsto\tau(t, z_0)$ is a loop representing an element $\alpha\in\pi_1(Z, z_0)$.

LEMMA 11.3. *Let $G=\pi_1(Z_0, z_0)$, let $\mu: G\to G$ be the map induced by m_τ and $\mathbf{Z}\times_\mu G$ be the semi-direct product. Then there is a unique isomorphism $h: \mathbf{Z}\times_\mu G\to\pi_1(Z, z_0)$ such that $h|G\times\{1\}$ is induced by the inclusion $Z_0\to Z$, and $h(\mathrm{id}, 0)=\alpha$.*

Proof. This is classical: the long exact sequence of the fibration gives a short exact sequence

$$1\to G\xrightarrow{i}\pi_1(Z, z_0)\to\mathbf{Z}\to 0;$$

and the map $n\mapsto n\alpha$ is a splitting, so that every element of $\pi_1(Z, z_0)$ can be written uniquely as $i(g)\cdot(n\alpha)$. (A representative for the product $x\cdot y$ is given by first traversing a path for x and then a path for y.) So to prove the lemma, all we need to show is

$$i(g)\cdot\alpha=\alpha\cdot i(\mu(g)).$$

Represent g by $\gamma: I\to Z_0$. Then the family of maps $\delta_t: [0, 2]\to Z$ given by

$$\delta_t(s)=\begin{cases}\tau(s, z_0), & 1\leqslant s\leqslant t\\ \tau(t, \gamma(s-t)), & t\leqslant s\leqslant t+1\\ \tau(s-1, z_0), & t+1\leqslant s\leqslant 2\end{cases}$$

is a homotopy between paths representing $i(g)\cdot\alpha$ and $\alpha\cdot i(\mu(g))$. Q.E.D. for Lemma 11.3

Lemma 11.3 can be applied to the fibration $\tilde{\Omega}\to\mathbf{C}-\tilde{D}$. First, note that the monodromy $m: \Omega(r)\to\Omega(r)$ considered in Chapter 10 does satisfy $m(P_0)=P_0$. Let

$$\mu: \pi_1(\Omega(r), P_0)\to\pi_1(\Omega(r), P_0)$$

be the automorphism induced by m. Let $\alpha\in\pi_1(\tilde{\Omega}, P_0)$ be the class represented by the loop given by the trivialization in Chapter 10. Remember that the boundary of $\tilde{\Omega}$ is canonically the set of $\{(\zeta, z)\,|\,|z|=|\zeta^3|\}$; the loop α is given by the parametrization $t\mapsto(re^{2\pi it}, r^3e^{6\pi it})$.

COROLLARY 11.4. *There is a unique isomorphism $\mathbf{Z}\times_\mu\pi_1(\Omega(r), P_0)\to\pi_1(\tilde{\Omega}, P_0)$ given by the canonical inclusion on $\pi_1(\Omega(r), P_0)$ and sending $(\mathrm{id}, 0)$ to α.*

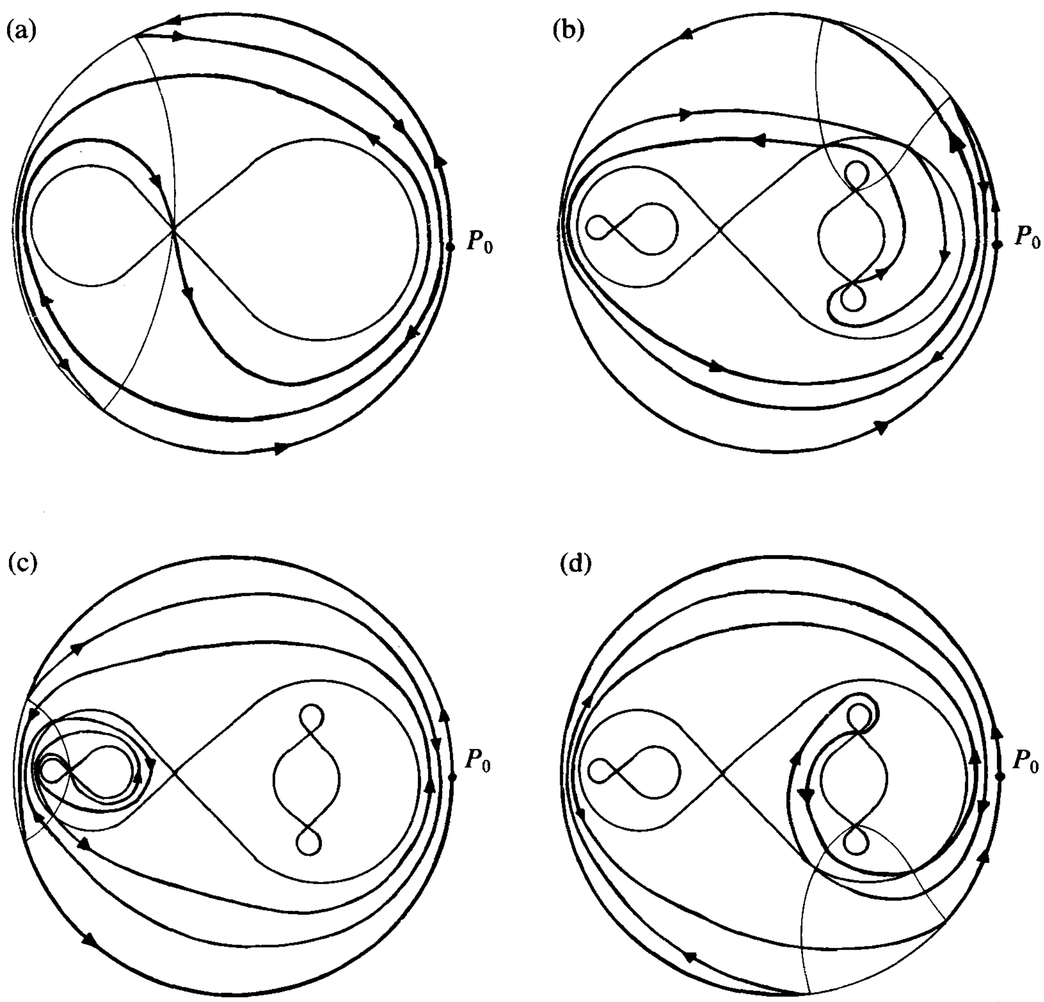


Fig. 11.2

11.4. Computing the monodromy of the parapattern bundle

We were a bit lighthearted above when we simply wrote "let

$$\mu: \pi_1(\Omega(r), P_0) \to \pi_1(\Omega(r), P_0)$$

be the automorphism induced by m". We do not know any simple way of computing the action of μ on the generators; in this section we will show how complicated this mapping μ actually is by computing a few examples. Note first that there is a homomorphism $\bar{\mu}$ induced by $\bar{m}$ such that $\bar{\mu}^2 = \mu$, which is a bit simpler than μ (without excess).

Our basic tool is the computation of $\bar{m}$ in terms of fractional Dehn twists. Recall Figure 10.1 from Section 10.4 in which we labelled the annuli of $\Omega(r)$ by the fraction of Dehn twist which occurs there. Using that figure, we obtain the following Figures 11.2 (a)–(d) for the images of $\gamma_{\{1/3,2/3\}}$, $\gamma_{\{1/9,2/9\}}$, $\gamma_{\{4/9,5/9\}}$ and $\gamma_{\{7/9,8/9\}}$.

We leave the diligent reader to verify that the loops drawn can be deformed as follows:

$$\bar{\mu}(\gamma_{\{1/3,2/3\}}) = \gamma_0^{-1}\gamma_{\{1/3,2/3\}}\gamma_0,$$

$$\bar{\mu}(\gamma_{\{1/9,2/9\}}) = \gamma_0^{-2}\gamma_{\{7/9,8/9\}}\gamma_0^2,$$

$$\bar{\mu}(\gamma_{\{4/9,5/9\}}) = \gamma_0^{-1}\gamma_{\{1/3,2/3\}}\gamma_0^{-1}\gamma_{\{4/9,5/9\}}\gamma_{\{1/3,2/3\}}^{-1}\gamma_0^2,$$

$$\bar{\mu}(\gamma_{\{7/9,8/9\}}) = \gamma_{\{1/3,2/3\}}^{-1}\gamma_0^{-1}\gamma_{\{1/9,2/9\}}\gamma_0^{-1}\gamma_{\{1/3,2/3\}}\gamma_0.$$

It is easiest to find these formulas by noticing that the class of a loop is entirely described by the order in which it crosses the descending rays from critical points of $H(r)$, and the descending ray of argument 1/2. Notice that depending on the order of the crossing we might have to insert γ_0 or γ_0^{-1} (compare with $\bar{\mu}(\gamma_{\{1/9,2/9\}})$ and $\bar{\mu}(\gamma_{\{7/9,8/9\}})$).

From the expressions for $\bar{\mu}$ above it follows that

$$\mu(\gamma_{\{1/3,2/3\}}) = \gamma_0^{-2}\gamma_{\{1/3,2/3\}}\gamma_0^2,$$

$$\mu(\gamma_{\{1/9,2/9\}}) = \gamma_0^{-2}\gamma_{\{1/3,2/3\}}^{-1}\gamma_{\{1/9,2/9\}}\gamma_0^{-1}\gamma_{\{1/3,2/3\}}\gamma_0^3,$$

$$\mu(\gamma_{\{4/9,5/9\}}) = \gamma_0^{-2}\gamma_{\{1/3,2/3\}}\gamma_0^{-1}\gamma_{\{1/3,2/3\}}\gamma_0^{-1}\gamma_{\{4/9,5/9\}}\gamma_{\{1/3,2/3\}}^{-1}\gamma_0\gamma_{\{1/3,2/3\}}^{-1}\gamma_0^3,$$

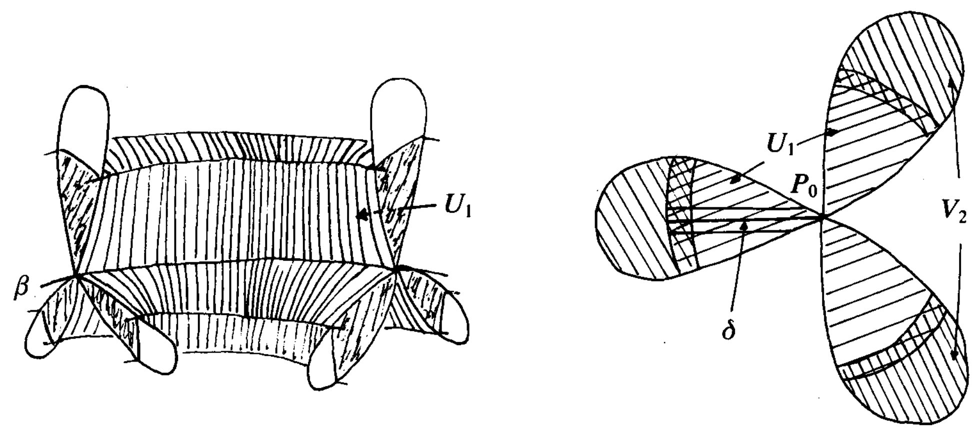
$$\mu(\gamma_{\{7/9,8/9\}}) = \gamma_0^{-1}\gamma_{\{1/3,2/3\}}^{-1}\gamma_0^{-1}\gamma_{\{7/9,8/9\}}\gamma_{\{1/3,2/3\}}\gamma_0^2.$$

This way of computing rapidly gets out of hand. Perhaps there is a better notation for such expressions.

11.5. The fundamental group of the quotient parapattern bundle

We now want to compute the fundamental group of Ω, which is obtained from $\tilde{\Omega}$ by the equivalence described in Section 7.6. More precisely, the points (ζ, ζ^3), $(j\zeta, \zeta^3)$ and $(j^2\zeta, \zeta^3)$ are identified. This "identification locus" can be parametrized by the path

$$t \mapsto (re^{2\pi it/3}, r^3e^{2\pi it}), \quad 0 \leqslant t \leqslant 1,$$



$\Omega(r) \cup \Omega(jr) \cup \Omega(j^2r)$

Fig. 11.3

which is a loop in Ω, starting at (r, r^3) and ending at (jr, r^3). Let β be the corresponding element of $\pi_1(\Omega, P_0)$.

PROPOSITION 11.5. *The group $\pi_1(\Omega, P_0)$ is freely generated by the image of $\pi_1(\tilde{\Omega}, P_0)$ and the element β, subject to the relation $\beta^3 = \alpha$.*

Proof. Let U_1 be a small neighborhood of β; by choosing this neighborhood "tubular" we will have $\pi_1(U_1, P_0) \cong \mathbf{Z}$, generated by β. Let V_2 be the complement in $\tilde{\Omega}$ of a smaller closed "tubular" neighborhood of β. The open set V_2 does not contain the base point; choose a path δ in $\Omega(r)$ connecting $\Omega(r) \cap V_2$ to P_0 and let U_2 be the union of V_2 and a small neighborhood of δ (see Figure 11.3). Such a U_2 is a deformation retract of $\tilde{\Omega}$, so the inclusion $U_2 \to \tilde{\Omega}$ is a homotopy equivalence.

Moreover, $U_1 \cap U_2$ is a bundle of discs over $\mathbf{C} - \bar{D}$ (with a handle, the part of the neighborhood of δ in U_1, attached). Thus $\pi_1(U_1 \cap U_2, P_0)$ is canonically isomorphic to $\mathbf{Z}$, and the inclusion $U_1 \cap U_2 \to U_2$ maps the canonical generator to α. On the other hand, the inclusion $U_1 \cap U_2 \to U_1$ maps the generator to β^3.

Now van Kampen's theorem tells us that $\pi_1(\Omega, P_0)$ is the amalgamated sum of the diagram

$$\begin{tikzcd} & \pi_1(U_2, P_0) \arrow[dr] & \\ \pi_1(U_1 \cap U_2, P_0) \arrow[ur] \arrow[dr] & & \pi_1(\Omega, P_0) \\ & \pi_1(U_1, P_0) \arrow[ur] & \end{tikzcd}$$

which translates into

$$\begin{tikzcd} & \pi_1(\tilde{\Omega}, P_0) \arrow[dr] & \\ \mathbf{Z} \arrow[ur, "1\mapsto\alpha"] \arrow[dr, "1\mapsto 3"'] & & \pi_1(\Omega, P_0) \\ & \mathbf{Z} \arrow[ur] & \end{tikzcd}$$

This is just another description of $\pi_1(\Omega, P_0)$ by generators and relations; β is the image of the generator of $\pi_1(U_1)$. Q.E.D. for Proposition 11.5

11.6. The fundamental group of the escape locus

Since $\mathscr{E}^+$ and $\mathscr{E}^-$ are symmetrical, their fundamental groups are isomorphic; let $\beta^\pm, \gamma_0{}^\pm$ be the corresponding elements in $\pi_1(\mathscr{E}^\pm, P_0)$. Note that $\beta^+=\beta^-=\beta$. Van Kampen's theorem tells us that the fundamental group $\pi_1(\mathscr{E}, P_0)$ is the amalgamated sum of the diagram

$$\begin{tikzcd} & \pi_1(\mathscr{E}^+, P_0) \arrow[dr] & \\ \pi_1(\mathscr{E}^+\cap\mathscr{E}^-, P_0) \arrow[ur] \arrow[dr] & & \pi_1(\mathscr{E}, P_0) \\ & \pi_1(\mathscr{E}^-, P_0) \arrow[ur] & \end{tikzcd}$$

So the only problem is to compute $\pi_1(\mathscr{E}^+\cap\mathscr{E}^-, P_0)$ and the maps induced by the inclusions.

PROPOSITION 11.6. *The group $\pi_1(\mathscr{E}^+\cap\mathscr{E}^-, P_0)$ is generated by $\gamma_0{}^+$ and β, subject to the relation $[\gamma_0{}^+, \beta^3]=1$. The element $\gamma_0{}^-$ satisfies $\gamma_0{}^+\beta\gamma_0{}^-=\beta^2$.*

Proof. To keep the notation almost reasonable, denote $T=\mathscr{E}^+\cap\mathscr{E}^-$ and let $\Delta_T\subset T$ the polynomials with a single critical point, i.e. those $P_{a,b}$ with $a=0$. Consider the mapping $\bar{F}: T\to\mathbf{R}_+\times(\mathbf{R}/\mathbf{Z})\times(\mathbf{R}/\mathbf{Z})$ defined by

$$\bar{F}(a,b) = (3\log s, \arg(\varphi_{a,b}(P_{a,b}(a))), \arg(\varphi_{a,b}(P_{a,b}(-a)))),$$

where $\log s=h_{a,b}(+a)=h_{a,b}(-a)$. This map is not quite a homeomorphism, it is a triple cover ramified along $\mathbf{R}_+\times\Delta$, where Δ is the diagonal of $\mathbf{R}/\mathbf{Z}\times\mathbf{R}/\mathbf{Z}$ as shown in Corollary 14.4 in [BH].

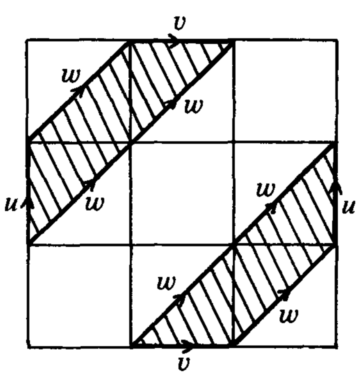


Fig. 11.4

To find a lifting of $\tilde{F}$ which is a homeomorphism, consider the subset $\tilde{S}\subset\mathbf{R}/\mathbf{Z}\times\mathbf{R}/\mathbf{Z}$ defined by $\tilde{S}=\{(\theta_1,\theta_2)|\, d(\theta_1,\theta_2)\geq 1/3\}$, and the equivalence relation

$$(\theta_1,\theta_2)\approx(\theta_1{}',\theta_2{}') \quad \text{if} \quad 3\theta_1=3\theta_2=3\theta_1{}'=3\theta_2{}'.$$

Set $S=\tilde{S}/\approx$, and consider the map $\pi\colon\tilde{S}\to\mathbf{R}/\mathbf{Z}\times\mathbf{R}/\mathbf{Z}$ given by $\pi(\theta_1,\theta_2)=(3\theta_1,3\theta_2)$, which clearly induces a map $S\to\mathbf{R}/\mathbf{Z}\times\mathbf{R}/\mathbf{Z}$, still called π. Note that π is a triple cover ramified above Δ.

In the obvious way of representing the torus by a square with sides identified, the region S is as pictured in Figure 11.4. Moreover, the equivalence relation above identifies only points of the boundary, and identifies them if they have the same first or second coordinate. For future reference, denote u, v and w respectively the loops in S represented by the segments from $(0,1/3)$ to $(0,2/3)$, $(1/3,0)$ to $(2/3,0)$ and $(0,1/3)$ to $(1/3,2/3)$ all of whose endpoints are identified in S to a single point p_0. The other segments of the diagonal lines marked in the figure are identified with w in S.

If $a\neq 0$ then

$$\zeta^+(a,b)=\lim_{z\to -2a}\varphi_{a,b}(z) \quad \text{and} \quad \zeta^-(a,b)=\lim_{z\to 2a}\varphi_{a,b}(z)$$

are well defined and $|\zeta^+|=|\zeta^-|=s$, and the mapping $\tilde{F}\colon T-\Delta_T\to\mathbf{R}_+\times\tilde{S}$ given by

$$\tilde{F}(a,b)=(\log s, \arg(\zeta^+(a,b)), \arg(\zeta^-(a,b)))$$

is an injective lifting of $\bar{F}$ on $T-\Delta_T$. It does not extend to Δ_T, since there are then no co-critical points, but it does induce a mapping $F\colon T\to\mathbf{R}_+\times S$.

Since both the mapping

$$\tilde{F}\colon T\to \mathbf{R}_+\times(\mathbf{R}/\mathbf{Z})\times(\mathbf{R}/\mathbf{Z}) \quad \text{and} \quad \mathrm{id}\times\pi\colon \mathbf{R}_+\times S\to \mathbf{R}_+\times(\mathbf{R}/\mathbf{Z})\times(\mathbf{R}/\mathbf{Z})$$

are triple covers ramified over $\mathbf{R}_+\times\Delta$, we see that F is a homeomorphism. Therefore

$$F_*\colon \pi_1(T,P_0)\to \pi_1(S,p_0)$$

is an isomorphism. The homomorphism F_* maps γ_0^+ to u, γ_0^- to v and β to w. From the description of S above, it is clear that u and w generate $\pi_1(S,p_0)$ and satisfy $[u,w^3]=1$, and that $uwv=w^2$. Q.E.D. for Proposition 11.6

Remark 11.7. In $\tilde{\Omega}$, whose boundary is a torus, the lift of γ_0^- starting at the base point is not closed: its endpoints are identified by the projection $\tilde{\Omega}\to\Omega$.

So a symmetrical description of the group $\pi_1(\mathscr{E}^+\cap\mathscr{E}^-,P_0)$ is: $\pi_1(\mathscr{E}^+\cap\mathscr{E}^-,P_0)$ is generated by 3 elements γ_0^+,γ_0^- and β, satisfying $\gamma_0^+\beta\gamma_0^-=\beta^2$ and $[\gamma_0^+,\beta^3]=[\gamma_0^-,\beta^3]=1$.

COROLLARY 11.8. *The group $\pi_1(\mathscr{E},P_0)$ is the amalgamated sum of the diagram*

$$\begin{tikzcd}
 & \pi_1(\mathscr{E}^+,P_0) \arrow[dr] & \\
\pi_1(\mathscr{E}^+\cap\mathscr{E}^-,P_0) \arrow[ur] \arrow[dr] & & \pi_1(\mathscr{E},P_0) \\
 & \pi_1(\mathscr{E}^-,P_0) \arrow[ur] &
\end{tikzcd}$$

where the mappings $\pi_1(\mathscr{E}^+\cap\mathscr{E}^-,P_0)\to\pi_1(\mathscr{E}^\pm,P_0)$ map $\beta\in\pi_1(\mathscr{E}^+\cap\mathscr{E}^-,P_0)$ to $\beta\in\pi_1(\mathscr{E}^\pm,P_0)$ and $\gamma_0^\pm$ to $\gamma_0\in\pi_1(\mathscr{E}^\pm,P_0)$.

Chapter 12. Polynomials of higher degree

Almost everything in this paper and the preceding one [BH] goes over to polynomials of arbitrary degree d, having two possibly multiple critical points. In this chapter we will outline how this goes, without proofs.

Our reason is not so much to generalize as to show the one thing which does not go through: Theorem 4.3 (a) and its consequences Theorem 5.2 and Theorem 9.1 (b) are false, or at least the proofs fail.

12.1. The global topology of parameter spaces

Choose d_1 and d_2 both positive with $d_1+d_2-1=d$, and set

$$P_{a,b}(z) = d\int_0^z (u-(d_2-1)\,a)^{d_1-1}(u+(d_1-1)\,a)^{d_2-1}\,du+b$$

with critical points $\omega_1=(d_2-1)a$ and $\omega_2=-(d_1-1)a$, of local degree d_1 and d_2 respectively. Clearly the case $d_1=d_2=2$ is the one we have been studying. We will say that such polynomials have *bidegree* (d_1, d_2).

We can define the connectedness locus

$$\mathscr{C}_{d_1,d_2} = \{(a, b) \in \mathbf{C}^2 |\text{ the Julia set of } P_{a,b} \text{ is connected}\}$$

for this family, and the function $H: \mathbf{C}^2 \to \mathbf{R}$ by

$$H(a, b) = \sup\{h_{a,b}(\omega_1), h_{a,b}(\omega_2)\}$$

which is continuous and vanishes exactly on $\mathscr{C}_{d_1,d_2}$.

The locus

$$\mathscr{S}_r = \{(a, b) | H(a, b) = \log r\}$$

is as before a 3-sphere for $r>1$, and $H: \mathbf{C}^2 - \mathscr{C}_{d_1,d_2} \to \mathbf{R}_+$ is a trivial fibration. This shows that $\mathscr{C}_{d_1,d_2}$ is cellular.

Remark 12.1. (a) Apparently our use of the word "cell-like" in [BH, Corollary 11.2] instead of cellular is inconsistent with previous usage [B]. Moreover, we claimed in the same corollary that the connectedness locus is contractible. One can only conclude weakly contractible.

(b) P. Lavaurs [L] has proved that the connectedness locus $\mathscr{C}_d$ is cellular in all degrees.

Further, we can write $\mathscr{S}_r = \mathscr{S}_r^1 \cup \mathscr{S}_r^2$, where

$$\mathscr{S}_r^i = \{(a, b) \in \mathscr{S}_r | H(a, b) = h_{a,b}(\omega_i)\} \quad \text{for} \quad i = 1, 2.$$

Theorem 14.1 of [BH] then generalizes as follows: recall that $(S^3; E_1, E_2)$ denotes the standard decomposition of S^3 into two linked solid tori, with $T=E_1 \cap E_2$ canonically homeomorphic to $S^1 \times S^1$, and Δ_T the diagonal in T.

THEOREM 12.2. *The triple $(\mathscr{S}_r; \mathscr{S}_r^1, \mathscr{S}_r^2)$ is homeomorphic to the d-fold cover of $(S^3; E_1, E_2)$ ramified along Δ_T.*

12.2. Patterns in higher degrees

As in Chapter 2 we can construct ζ-patterns of bidegree (d_1, d_2) for any $d_1, d_2 \geqslant 2$. For each ζ-pattern of depth $N>0$ the mapping $\pi_R: R \to s_N(R)$ will be a d-fold cover ramified at ω_R and with π_R of local degree d_1 at ω_R.

In the construction of patterns Lemma 2.2 was essential. The analogous lemma still holds with essentially the same proof:

LEMMA 12.3. *Let a and b be two points in $X=\mathbf{C}$. Then there exists a connected d-fold cover $\pi: Y \to X$ ramified above a and b such that $\pi^{-1}(a)$ and $\pi^{-1}(b)$ each contain precisely one critical point, of local degree d_1 and d_2 points respectively. If $\pi_Y: Y \to X$ and $\pi_Z: Z \to X$ are two such covers, there exists a unique covering homeomorphism $Y \to Z$.*

With this lemma the complete construction of the pattern tree in Section 2.2 goes over exactly as in Chapter 2, except for the construction of $i_{R_1}: R_0 \to R_1$, which requires a small modification. In this case there is no unique co-critical point if $d_2>2$, but it is possible to distinguish one of the d_2-1 other (co-critical) inverse images of $\pi_R{}^{-1}((\pi_R(\omega_R))$ as follows.

The inverse image under π_{R_1} of the circle Γ_0 of radius $|\zeta|^d$, which is contained in R_0, is a graph Γ_1 in R_1 consisting of d_1-1 loops which cover Γ_0 with degree 1 and one loop $\Lambda \subset \partial C_1(R)$ which covers Γ_0 with degree d_2. All these loops touch at the point ω_R. The co-critical points all lie on Λ, hence are circularly ordered. Let us choose ω'_{R_1} to be the first one after ω_{R_1} for this order.

Now analogously to Section 2.2 we can define $i_{R_1}: R_0 \to R_1$ as the unique covering isomorphisms $R_0 \to \pi_{R_1}^{-1}(R_{-1})$ satisfying $\lim_{z \to \zeta} i_{R_1}(z) = \omega'_{R_1}$.

There are precisely $d-1$ patterns isomorphic to a given one $R \in \mathcal{P}(\zeta)$; they lie in $\mathcal{P}(\zeta e^{2\pi ik/(d-1)})$ where $k=0,1,\ldots,d-2$. The isomorphisms are constructed as in Section 2.7 by repeated liftings of the maps $z \mapsto e^{2\pi i/(d-1)}z$, which commute with $z \mapsto z^d$.

As in Section 2.8 we can construct pattern bundles. The adaptations of the statements in 2.8 are left to the reader.

All the statements in Chapter 3 go through in the obvious way.

12.3. Ends of patterns and tableaux

As in Chapter 4 for each $R \in \mathcal{P}_\infty(\zeta)$ we can construct tableaux satisfying precisely the same three tableau rules. Theorem 4.2 and Theorem 4.3 (b) and (c) are still true, but the

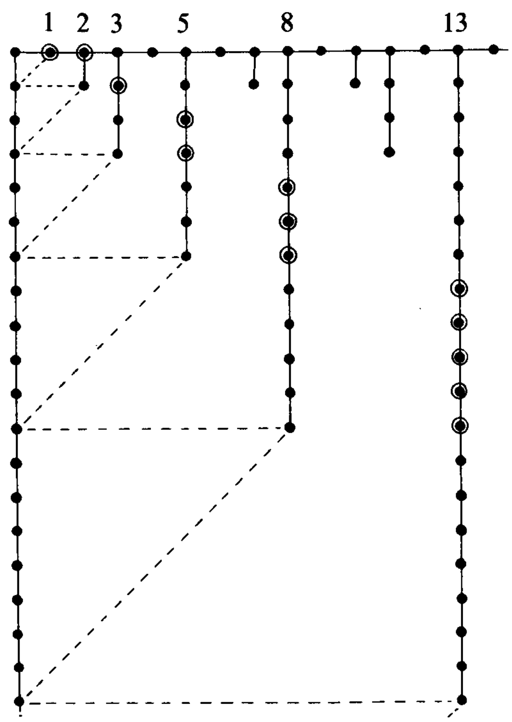


Fig. 12.1. The Fibonacci grid.

statement corresponding to Theorem 4.3 (a) *fails* if $d_2>2$. More precisely, in the proof of Lemma 4.5 we sum a geometric series with ratio 1/2; this ratio becomes $1/d_2$ in general and the proof simply does not go through. In fact, the result is false, as shown in the example below.

Example 12.4. Consider the *Fibonacci* marked grid in Figure 12.1, which can be described more formally as the marked grid whose critical staircase has tips (b_k, a_k) where the a_k are the Fibonacci numbers $a_1=2$, $a_2=3$, $a_3=5, \ldots$ with $a_{k+1}=a_k+a_{k-1}$ and the b_k satisfy the recursion relation $b_{k+1}=a_k+b_k$ starting with $b_1=1$.

This critical marked grid satisfies the tableau rules, and its originators are as marked: in particular there is exactly one in every row except the 0th which contains exactly 2. In this case the computation analogous to Lemma 4.5 gives

$$\operatorname{mod} N(c) = \sum_{j=0}^{\infty} \operatorname{mod}(C_j) = \operatorname{mod}(C_0) + \frac{1}{d_2-1} \sum_{\substack{(l,k) \\ \text{originator}}} \operatorname{mod}(A_{l,k}(c))$$

$$= \operatorname{mod}(C_0) + \frac{1}{d_2-1}\left(\operatorname{mod}(C_0) + \sum_{j=0}^{\infty} \operatorname{mod}(C_j)\right).$$

For $d_2>2$ this gives the following formula for the modulus of the critical nest

$$\operatorname{mod} N(c) = \frac{d_2}{d_2-2}\operatorname{mod}(C_0).$$

Hence the critical end is convergent and non-periodic.

Remark 12.5. There is another way of understanding this example. Recall from Theorem 4.3 (a) the definition of the function t. For this tableau, the set $t^{-1}(n)$ has exactly two elements for every n, so that for any k,

$$\sum_{\operatorname{gen}(j)=k+1} \operatorname{mod}(C_j) = \frac{2}{d_2}\sum_{\operatorname{gen}(j)=k} \operatorname{mod}(C_j).$$

Therefore the series whose kth term is the sum of the moduli of generation k converges.

12.4. Julia sets of polynomials of higher degree

In light of the results above, we do not get the analogue of Theorem 5.2 in degree greater than 3. The best we can do without new techniques are the following results:

THEOREM 12.6. *Let P be a polynomial of bidegree (d_1, d_2). Suppose ω_1 escapes to ∞ and $d_2=2$. Then*

(a) *K_P is a Cantor set if and only if the critical component of K_P is not periodic, and*
(b) *if K_P is a Cantor set, then it is of measure* 0.

THEOREM 12.7. *Let P be a polynomial of bidegree (d_1, d_2). If ω_1 escapes to ∞ and the critical component of K_P is non-recurrent, then K_P is a Cantor set of measure* 0.

Proofs. The proof of Theorem 12.6 is exactly the same as for Theorem 5.2 and Theorem 5.9; for in that case Theorem 4.3 (a) does hold: the ratio of the geometric series occurring in Lemma 4.5 is 1/2.

As mentioned in the introduction to Chapter 5, there is an easier proof of Theorem 4.3 (a) in the case of non-recurrence: in that case the annuli of the critical nest have moduli which are bounded below. This still holds in higher degrees. Theorem 12.7 follows. Q.E.D. for Theorems 12.6 and 12.7

12.5. Parapatterns in higher degree

All the constructions and theorems in Chapter 7, 8, 9 go through to the case under consideration. Without repeating the statements, we want to isolate the following result, generalizing Remarks 8.3 and 9.4:

PROPOSITION 12.8. *There exists a polynomial of bidegree (d_1, d_2) realizing any pattern of bidegree (d_1, d_2).*

12.6. Ends of parapatterns and Julia sets

Theorem 5.2 proves the following folk-lore conjecture in degree 3.

CONJECTURE 12.9. *Let P be a polynomial and let Ω_b be the set of critical points with bounded orbits. The Julia set J_P is a Cantor set if and only if for all $\omega \in \Omega_b$ and for all $n>0$ $P^{\circ n}(K_P(\omega)) \neq K_P(\omega)$, where $K_P(\omega)$ is the component of K_P containing ω.*

Fatou and Julia knew this when $\Omega_b = \emptyset$ and when $\Omega_b = \Omega$. Furthermore, we know from the theory of polynomial-like mappings that if there exists a critical point $\omega \in \Omega_b$ and an $n>0$ such that $P^{\circ n}(K_P(\omega)) = K_P(\omega)$ then the Julia set J_P is not a Cantor set. But we do not know in general whether the nonperiodicity condition of critical components of K_P is sufficient to guarantee that the Julia set is a Cantor set. We know that the proof given in this paper does not go through.

More specifically, by the analogue of Theorem 4.2 (a), the critical marked grid in Example 12.4 is the marked grid of some pattern of infinite depth, and by Proposition 12.8, there is a polynomial with this pattern. In fact, the careful reader will find going through the proof of Theorem 4.2 (a) with this particular critical marked grid in mind that there are infinitely many choices to make, hence infinitely many patterns and infinitely many polynomials of this sort.

For these polynomials, the critical component of K_P is recurrent and non-periodic; and the nest is convergent. *We do not know whether this component is a point.* By Sullivan's no wandering domains theorem [S], the component has empty interior, and in fact has measure zero (this last fact was pointed out to us by Curt McMullen).

Even though such a component has measure zero, and even if the critical component is a point so that the Julia set is a Cantor set, the Julia set might have positive measure. The proof of Theorem 5.9 does not extend to this case.

Furthermore, we know nothing about the component of $\mathscr{B}(\zeta)$ in the parameter space corresponding to such polynomials. Sullivan's theorem does not apply to parameter space, and as far as we know such a component might have interior, corresponding to some P-invariant Beltrami form carried by the Julia set (which would have to have positive measure for this to occur).

We cannot settle Conjecture 12.9, but feel that figuring out whether it is true for these examples is an interesting problem.

For cubics the subset of $\bar{\mathscr{B}}$ corresponding to polynomials with critical tableaux equal to the Fibonacci grid forms a fiber bundle over $\mathbf{C}-\bar{D}$ with fibers a Cantor set. It can be shown to be connected and hence homeomorphic to the dyadic solenoid (Proposition 10.7 and Remark 10.10).

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